# SURGE TANK PROBLEMS

*AN INVESTIGATION OF SURGE TANK REGULATION DETERMINING BY GRAPHICAL AND ANALYTICAL METHODS PROPER SOLUTIONS OF PROBLEMS CREATED BY LONG PIPE LINES.*

BY

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AUTHORIZED TRANSLATION

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AN INVESTIGATION OF SURGE TANK REGULATION DETERMINING BY GRAPHICAL AND ANALYTICAL METHODS PROPER SOLUTIONS OF PROBLEMS CREATED BY LONG PIPE LINES.

By PROF. FRANZ. PRASIL,

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IN case the main conduit leading to a turbine is comparatively long, beginning at the intake on a comparatively light slope (canal, tunnel or low-pressure pipe) and concluding in a short steep slope leading directly to the turbines, (high-pressure pipe or penstock), it is common practice to construct at the junction an open tank, called the surge tank, by which the main conduit is separated into two parts, in which with a constant discharge of the turbines the flow in the conduit is constant, and with a variable turbine discharge a variable conduit flow results. In the first case, the water surface in the surge tank is lower than the water surface at the intake by an amount dependent upon the flow in the low-pressure pipe. This difference of head is determined by the friction head in the main conduit. In an increment of time just as much water flows into the surge tank as flows out of it through the penstocks to the turbines.

In the second case, the inertia of the moving mass in the main conduit prevents this equality of inflow and outflow. The water surface in the surge tank has variable heights, that is, it rises or falls above and below the elevation due to the steady flow. The extent of this fluctuation of the water surface depends upon the dimensions of the main conduit, the amount of flow, and also on the size and form of the surge tank. In this case, if an overflow or an excessive lowering or too large fluctuations of the water surface are to be prevented, the surge tank must be dimensioned according to the area and length of the main conduit and according to the inflow and outflow. In this article the problems leading to the determination of the dimensions will be discussed, and the methods, partly analytical and partly graphical, involved in the solution of these problems, will be developed upon the assumption that the main conduit from the intake to the surge tank consists of a pipe or tunnel running full under pressure.

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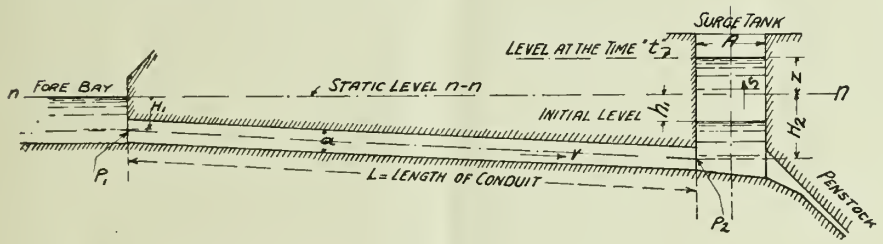


Fig. 1.

## I. Introduction.

The investigation is developed (see Fig. 1) under the following additional assumptions:

(1) The intake is provided with a spillway whose dimensions are such that the elevation $n-n$ in the forebay may be considered as constant during the period under investigation.

(2) The sectional area of the main conduit is constant.

(3) The volume of the conduit, compared to the volume of the surge tank and compared to that part of the volume of the forebay which is affected, is so large that the influence of both of these masses of water toward decreasing the flow may be neglected.

(4) Elastic and temperature conditions are neglected.

In the derivations of the formulæ, the following abbreviations are used:

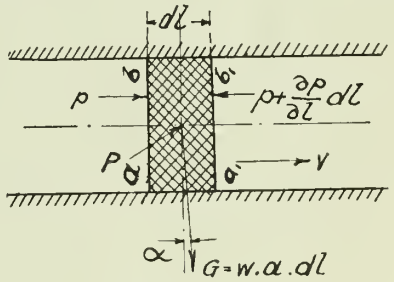


Fig. 2.

$L$ = Length of main conduit in feet.

$a$ = Sectional area of main conduit in square feet.

$p$ = Wetted perimeter of main conduit in feet.

$v$ = Velocity of water in main conduit in feet per second at the time "$t$".

$v_1$ = Normal velocity of water in the main conduit in feet per second, during the period of steady flow.

$v_2$ = Initial velocity in the main conduit in feet per second, at the time $t = o$.

($v$, $v_1$ and $v_2$ are average values and are assumed as constant throughout the entire length of the main conduit).

$H_1$ = Vertical distance from the water surface $n-n$ in the forebay to the centre of gravity of the entrance to the main conduit.

$H_2$ = Vertical distance from the water surface $n-n$ in the forebay to the centre of gravity of the entrance to the surge tank.

$Q$, $Q_1$, $Q_2$ = Volumes of water, corresponding to time $t$, the time during steady flow, and for time $t = o$ (beginning).

$h$, $h_1$ and $h_2$ = Friction heads corresponding to the pipe dimensions and velocities $v$, $v_1$ and $v_2$.

$z$ = Vertical distance from the elevation $n-n$, of the water surface in the surge tank at the time $t$, taken positive above $n-n$ and negative below $n-n$.

$A$ = Sectional area of surge tank in square feet for the elevation determined by $z$, so that in general $A$ is a function of $z$.

$s$ = Velocity of the water surface elevation in the surge tank, positive when rising, negative when falling, as an average value assumed constant in the section $A$.

$q$ = Volume of discharge through the penstock at the time $t$, in cubic feet per second.

$c = \frac{q}{A}$ = the discharge velocity of the volume $q$, in feet per second, with respect to $A$.

More notations will be introduced during the course of the investigations.

**II. Derivation of Principal Equations.**—Referring to Fig. 2, the distance between two adjacent cross-sections of the main conduit $= dl$, and therefore $\frac{w.a.dl}{g}$ = mass of the water between the two sections ($w$ = weight of cubic unit of water, $g$ = acceleration due to gravity).

At the left side cross-section at the time $t$, there exists the pressure $p$ in pounds per square foot, an average value, assumed constant for the entire cross-section. In the right, the pressure is $p'$ at the same time $t$. The whole of $p'$ is in general different from that of $p$, by the amount $dp$. The pressure $p$ is a function of the location of the cross-section, that is to say, depends upon $l$; ($l$ = the distance of the left side cross-section from the entrance of the main conduit) and depends further upon the time $t$ (the flow varies with the time).

$$dp = \frac{\delta p}{\delta l} dl + \frac{\delta p}{\delta t}.dt \quad - \quad - \quad (1)$$

Since $p$ and $p'$ occur at the same time, $p'$ is different from $p$ by the difference due to the distance between the cross-sections, and therefore in the foregoing formula the differential $dt$ of the time $= o$. Therefore, $dp = \frac{\delta p}{\delta l}.dl$ (2).

The following forces act upon an element of unit mass, in the direction of flow, that is, in the direction of $v$:

1st. The weight component $P_1 = w.a.dl \sin \alpha$ (3) where $\alpha$ = the inclination of the axis of the main conduit from the horizontal, $dl.\sin \alpha = dH$ = vertical distance between the centres of gravity of the sections $a\,b$ and $a_1 b_1$.

$$P_1 = w.a.dH \quad - \quad - \quad (4)$$

2nd. The difference between the reactions due to the pressure $p$ and $p + \frac{\delta p}{\delta l}.dl$ which is

$$P_2 = p.a - (p + \frac{\delta p}{\delta l}.dl)\, a = - a \frac{\delta p}{\delta l} dl \quad - \quad (5)$$

Contrary to the direction of motion the friction exerts a force. Let $K$ be the amount of the friction in terms of the velocity $v$ and we get $P = - wa.dl.K$ (6) where $K$ represents a pure number with respect to its dimension.

According to the general fundamental law: mass × acceleration = acting force, it results (as $\frac{dv}{dt}$ is the acceleration with respect to the velocity $v$ at the time $t$) that,

$$m.\frac{dv}{dt} = P_1 + P_2 - P \quad - \quad - \quad - \quad (7)$$

$$\frac{w.a.dl}{g} = w.a.dH - a\frac{\delta p}{\delta l} dl - w.a\,dl.K \quad (8)$$

$$\frac{dl}{g}.\frac{dv}{dt} = dH - \frac{1}{w}\frac{\delta p}{\delta l} dl - K\,dl \quad - \quad - \quad (9)$$

The velocity $v$ has the same value at the time $t$ in the whole length of the main conduit. The case is therefore the same for $\frac{dv}{dt}$ and $K$. Therefore, if we integrate between the limiting values $l = o$ and $l = L$ or $H = H_1$ and $H = H_2$ relative to a motion from the intake to the main conduit, we get the following:

$$\frac{L}{g}\frac{dv}{dt} = H_2 - H_1 - \frac{1}{w}\int_0^L \frac{\delta p}{\delta l} dl - KL \quad (10)$$

In the integral $\int_0^L \frac{\delta p}{\delta l}.dl$, $p$ is, as demonstrated, a function of $t$ and $l$. But as the integration relates to the condition at a certain time $t$, $t$ is to be considered as a constant and therefore

$$\int_0^L \frac{\delta p}{\delta l} dl = p_2 - p_1 \quad - \quad - \quad (11)$$

where $p_2$ = pressure at the entrance to the surge tank $p_1$ at the entrance to the main conduit, $KL$ is nothing other than the friction head $h$ for the entire main conduit at the time $t$. Now, we may say that

$$\frac{p_2}{w} = H_2 + z + \frac{p_0}{w}; \quad \frac{p_1}{w} = H_1 + \frac{p_0}{w} \quad (12)$$

in which $\frac{p_0}{w}$ equals the water column equivalent to the atmospheric pressure, and the following equation results:

$$\frac{L}{g}.\frac{dv}{dt} + z + h = o \quad - \quad - \quad (13)$$

Having assumed a flow in the main conduit from the intake to the surge tank, we consider now the flow as reversed (from the surge tank to the intake) and keep the direction for the measurement of the length $l$ the same, then we must consider that the friction now acts in the direction of an increase of $l$, contrary to the first instance. In this case we get the equation

$$\frac{L}{g}\frac{dv}{dt} + z - h = o \quad - \quad - \quad (14)$$

For any case we may combine both equations into a fundamental formula:

$$\frac{L}{g} \cdot \frac{dv}{dt} + z \pm h = o \quad - \quad - \quad (15)$$

in which the plus refers to the velocity from intake to surge tank and the minus sign for the reverse movement. If we now introduce for $h$ a function, the value of which has the same sign as $v$, then the double sign may be dropped and the whole movement may be represented by the equation:

$$\frac{L}{g} \cdot \frac{dv}{dt} + z + h = o \quad - \quad - \quad (16)$$

A second fundamental equation may be developed from the condition of continuous flow, that is to say, the volume of water which flows to the surge tank in the time element $dt$ must be equal to the sum of the changes of volume in the surge tank and in the volume flowing out of the tank at the same time ∴

$$v.a.dt = s\,A\,dt + q.dt.$$

$$a.v = As + q = A(s + c) \quad - \quad (17)$$

The questions which are of principal interest are:

1st. How is the movement of the water surface in the surge tank with regard to time affected by given dimensions of the main conduit and the surge tank and given conditions as to overflow?

2nd. What dimensions must the surge tank have to agree with given conditions as to the main conduit and outflow, in order that the fluctuation of water level does not exceed certain amounts, determined by local conditions? The following cases will be investigated:

(a) Sudden partial or entire cessation of the outflow.

(b) Sudden starting of the outflow.

(c) Gradual cessation, or starting, variable outflowing conditions; and

(d) Influence of a spillway built in the surge tank.

The formulæ will be developed in the first case for a constant sectional area of surge tank, and under the assumption that $h$ is proportional to $v$, i.e., $h = n.v$.

It will be shown that it is possible to investigate all cases according to a uniform method, analytically and graphically, with the assistance of the well-known theory of damped and forced undulations. The inaccuracies which result because $h$ is proportional to $v^2$ and the corrections which must be applied to the results of the first method will be shown in the final study. With the simplifying assumptions already mentioned, we get from equation (17)

$$v = \frac{A}{a}(s + c) \qquad \frac{dv}{dt} = \frac{A}{a}\left(\frac{ds}{dt} + \frac{dc}{dt}\right) \quad - \quad (18)$$

This applied in equation (16) and the whole equation divided by

$$T^2 = \frac{L}{g} \cdot \frac{A}{a} \quad (19), \text{ we get } \frac{ds}{dt} + \frac{dc}{dt} + \frac{z}{T^2} + \frac{h}{T^2} =$$

$$\frac{ds}{dt} + \frac{n.A}{T^2.a} \cdot s + \frac{z}{T^2} + \frac{nA}{T^2 a} \cdot c + \frac{dc}{dt} = o \quad (20)$$

If we introduce as an abbreviation:

$$T_o = \frac{T^2.a}{n.A} \quad - \quad - \quad (21)$$

and consider that

$$s = \frac{dz}{dt} \qquad \frac{ds}{dt} = \frac{d^2z}{dt^2} \quad - \quad - \quad (22)$$

then follows the principal equation:

$$\frac{d^2z}{dt^2} + \frac{1}{T_o} \cdot \frac{dz}{dt} + \frac{z}{T^2} + \frac{c}{T_o} + \frac{dc}{dt} = o \quad (23)$$

The values $n$ and $T = \sqrt{\frac{A\,L}{a.g}}$

$$\text{and } T_o = \frac{T^2}{n} \cdot \frac{a}{A} = \frac{L}{n.g} \quad - \quad (24)$$

are times with regard to their dimensions. Starting with this principal equation, we may investigate the different cases as follows:

## III. Special Cases.

### *Case A.—Sudden Shut-down.*

Preceding a shut-down, $Q_1$ cubic feet per second flows out of the surge tank. During the normal condition in the main conduit, the velocity $= v_1 = \frac{Q_1}{a}$ wherefore $q = Q_1$. The water surface in the surge tank is $h_1$ feet lower than the static level $n - n$. The time $t$ is measured from the beginning of the shut-down, therefore from $t = o$, $q$ becomes $\epsilon q$ if $\epsilon$ is the proportion of the steady flow subsequent to the shut-down in relation to the flow preceding the shut-down.

After the sudden shut-down, the following phenomena occur in the surge tank: The water surface rises with variable velocity until it reaches a maximum height. When the highest elevation is reached, the reverse movement occurs. The velocity increases as the water level recedes, then decreases until the lowest level is reached, after which an ascending movement occurs but to a somewhat less height than before, and so on, until the normal conditions with the constant flow $\epsilon.Q_1$ become established. The movement of the water level belongs in the category of the damped oscillations.

(1) Analytical Investigation.—The formula (23) for this case, under the condition that

$$c = \frac{q}{A} = \frac{\epsilon Q_1}{A} = \epsilon c_1 = \text{a constant } \left(\frac{dc}{dt} = o\right) \quad (25)$$

becomes

$$\frac{d^2z}{dt^2} + \frac{1}{T_o}\frac{dz}{dt} + \frac{z}{T^2} + \epsilon\frac{c_1}{T_o} = o \quad - \quad (26)$$

Since $h_1 = n.v_1 = n\,c_1 \frac{A}{a} = c_1 \frac{T^2}{T_o}$, it follows that

$$\frac{\epsilon c_1}{T_o} = \frac{\epsilon h_1}{T^2} \quad (27)$$

If we introduce $z = y - \epsilon h_1$, therefore $\frac{dz}{dt} = \frac{dy}{dt}$;

$$\frac{d^2z}{dt^2} = \frac{d^2y}{dt^2} \quad - \quad - \quad (28)$$

from (26) follows a differential equation of the second order

$$\frac{d^2y}{dt^2} + \frac{1}{T_0} \cdot \frac{dy}{dt} + \frac{y}{T^2} = o \quad - \quad (29)$$

This is a linear differential equation of the second order with constant coefficients. For the solution we say:

$$r^2 + \frac{1}{T_0} r + \frac{1}{T^2} = o$$

The solution of this quadratic equation will give us the values of the differential equation:

$$r = -\frac{1}{2T_0} \pm \sqrt{\frac{1}{(2T_0)^2} - \frac{1}{T^2}}$$

We know that according to the theory of the linear differential equations three solutions may be found.

If the radical is positive or zero, the solution of the differential equation represents a non-periodical function, that is:

$$y = R_1 . e^{-\frac{t}{2T_0}} + R_2 t e^{-\frac{t}{2T_0}}$$

$$\text{if } \frac{1}{T_1^2} = \frac{1}{T^2} - \frac{1}{(2T_0)^2} = o \quad (30a)$$

$$y = (R_1 e^{+\frac{t}{T_1}} + R_2 e^{-\frac{t}{T_1}}) e^{-\frac{t}{2T_0}}$$

$$\text{if } \frac{1}{T_1^2} = \frac{1}{T^2} - \frac{1}{(2T_0)^2} = \text{negative} \quad (30b)$$

But,

$$y = R e^{-\frac{t}{2T_0}} \cdot \sin(\beta + t/T_1)$$

$$\text{if } \frac{1}{T_1^2} = \frac{1}{T^2} - \frac{1}{(2T_0)^2} = \text{positive} \quad (30c)$$

represents a damped harmonic.

These three conditions, due to the different values of the radical of the quadratic equation can be written also, substituting for $T$ and $T_0$ the values of equations 24:

$$\frac{1}{T_1^2} \text{ becomes zero if } T = 2T_0 \text{ or } \frac{A}{a} = \frac{4L}{n^2 g} \quad (31a)$$

$$\frac{1}{T_1^2} \text{ becomes negative if } T > 2T_0 \text{ or } \frac{A}{a} > \frac{4L}{n^2 g} \quad (31b)$$

$$\frac{1}{T_1^2} \text{ becomes positive if } T < 2T_0 \text{ or } \frac{A}{a} < \frac{4L}{n^2 g} \quad (31c)$$

As mentioned before, equations 30*a* and 30*b*, with the conditions of 31*a* and 31*b*, represent non-periodical function, that is:

Form 1 is the expression for a damped oscillation;

Forms 2 and 3 represent non-periodical movements, i.e., a transition from one quiescent level to another without any oscillations.

As will be seen later from an example, $n$ has in most cases a value which lies between 2 and 1 seconds; therefore, the condition for a non-periodic water level fluctuation is:

$$\frac{A}{a} \text{ is equal to or greater than } \frac{L}{35}.$$

Now, for $L = x$ miles

$A$ should be equal to or greater than $150.x.a$.

This case may well occur when a pond is used as the surge tank. With artificially constructed surge tanks $\frac{A}{a}$ is considerably smaller; the following investigations are therefore limited to the first form of oscillations. As $z = y - \epsilon h_1$, equation (30) may be written

$$z = -\epsilon h_1 + R e^{-\frac{t}{2T_0}} \sin(\beta + t/T_1) \quad - \quad (32)$$

and the differentiation with respect to $t$ gives (since $\frac{dz}{dt} = s$)

$$s = Re^{-\frac{t}{2T_0}} \left\{ \frac{1}{2T_0} \cdot \frac{2T_0}{T_1} \cos(\beta + t/T_1) - \sin(\beta + t/T_1) \right\} \quad (33)$$

If we substitute $tg\gamma = \frac{2T_0}{T_1}$ and consider that

$$\frac{1}{T_1} = \frac{1}{T^2} - \frac{1}{4T_0^2} \quad - \quad - \quad (34)$$

we get:

$$s = \frac{R}{T} \cdot e^{-\frac{t}{2T_0}} \sin(\gamma - \beta - t/T_1) \quad (35)$$

and the integration constants $R$ and $\beta$ are determined from the initial conditions, i.e., from the location and the condition of movement of the water level in the surge tank at the time $t = o$.

It will be noted from the description of the phenomena for that case that for the shut-down and in the very moment of it, the water surface is at the distance $h_1$ under the elevation $n - n$. Therefore, for $t = o$, $z = z_0 = -h_1$. The initial value for $s = s_0$ at the time $t = o$, must be assumed as

$$s_0 = \frac{Q_1 - \epsilon Q_1}{A} = (1 - \epsilon) c_1 \quad - \quad (36)$$

This assumption does not only consider a sudden shut-down, but assumes also a sudden beginning of a

uniformly distributed velocity in the water surface of surge tank. This assumption is true for the limiting cases because a short duration of the shut-down does not appreciably affect the results, as we shall see from the investigation of the influence of a shut-down of short duration. Here follow the equations for the determination of the integration constant when $t = o$:

$$R \sin \beta = -(1-\epsilon)\, h_1 \quad - \quad - \quad (37)$$

$$R \sin (\gamma - \beta) = +(1-\epsilon)\, c_1 T \quad - \quad (38)$$

The latter equation may be transformed to:

$$R \cos \beta = (1-\epsilon)\left[\frac{T_0}{T^2} - \frac{1}{2\,T_0}\right] h_1\, T_1 \quad (39)$$

By introducing the values in the bracketed term, we find that this term is proportional to the difference

$$\left(\frac{a}{A} - \frac{n^2 g}{2\,L}\right) \quad - \quad - \quad (40)$$

The difference is therefore positive if $\frac{A}{a}$ is less than $\frac{2\,L}{n^2 g}$, i.e., with the same assumptions for $n$ and $g$ as before:

$$A < 17.5 . x . a$$

This has influence on the determination of the value of $\beta$, as in the case mentioned sin $\beta$ is negative and with $\frac{A}{a} < \frac{2\,L}{n^2 g}$ the cos $\beta$ becomes positive, i.e., $\beta$ must lie in the fourth quadrant. If $\frac{4\,L}{n^2 g} > \frac{A}{a} > \frac{2\,L}{n^2 g}$, the bracketed expression and $\therefore$ cos $\beta$ must be negative, $\beta$ lies in the third quadrant. For the last formula, we may find for the determination of $R$ and $\beta$ the equations:

$$R = (1-\epsilon)\, h_1 \frac{T_1 T_0}{T^2} \quad - \quad - \quad (41)$$

$$tg\beta = -\frac{1}{T_0/T_1 - \tfrac{1}{4}\, T_1/T_0} \quad - \quad (42)$$

41 is obtained by addition of the squared equations 37 and 39;

42 is the quotient of $\frac{\text{equation 37}}{\text{equation 39}}$

We see that the amount of shut-down has influence on the size of $R$, but not on the size of $\beta$. With respect to the occurrence of the movements (see equations 32 and 35) we find that the movement is a damped oscillation with a duration of $\delta$ secs $= 2\pi T_1$.

Maximum and minimum values of $z$ or $y$ occur when $\frac{dz}{dt} = o$.

This is the case when sin $(\gamma - \beta - t/T_1) = o$. See Equation 35.

$$\frac{t}{T_1} = \gamma - \beta = \gamma - \beta + \pi = \gamma - \beta + 2\pi = \text{etc.} \quad (43)$$

This value used for equation 32 gives the following maximum values:

$$z\,\max_1 = -\epsilon h_1 + Re^{-\frac{T_1}{2\,T_0}(\gamma - \beta)} \sin \gamma$$

$$z\,\max_2 = -\epsilon h_1 + Re^{-\frac{T_1}{2\,T_0}(\gamma - \beta + 2\pi)} \sin \gamma, \text{ etc.} \quad (44)$$

and the following minimum values:

$$z\,\min_1 = -\epsilon h_1 - Re^{-\frac{T_1}{2\,T_0}(\gamma - \beta + \pi)} \sin \gamma$$

$$z\,\min_2 = -\epsilon h_1 - Re^{-\frac{T_1}{2\,T_0}(\gamma - \beta + 3\pi)} \sin \gamma, \text{ etc.} \quad (45)$$

Due to $z = y - \epsilon h_1$

$$\frac{y\,\max_2}{y\,\max_1} = \frac{y\,\max_3}{y\,\max_2} = \ldots\ldots\ldots$$

$$= e^{-\frac{T_1}{2\,T_0}.2\pi} = \frac{y\,\min_2}{y\,\min_1} = \frac{y\,\min_3}{y\,\min_2} = \ldots\ldots \quad (46)$$

The amplitude of this oscillatory motion is decreasing.

If $t$ = infin. $z$ becomes $-\epsilon h_1$
$s$ becomes zero

(2) Graphical Demonstration.—For the graphical demonstration of the velocity, a very simple construction may be used. We may divide the equation 32 into two equations which, using $\phi = t/T_1$, may be written:

$$B = Re^{-\frac{T_1}{2\,T_0}.\phi} \qquad y = B \cdot \sin(\beta + \phi) \quad (47)$$

The former gives in the polar system of co-ordinates (Fig. 3) ($\phi$ in radians) with $\beta$ as the vector radius a logarithmic spiral with the slope $\alpha = -T_1/2T_0$.

The second equation may be demonstrated in the rectangular system of co-ordinates, where $y$ and therefore $z$ is a function of $\phi$, if we consider the axis of abscissæ in the continuation of this ray of the polar system, which is turned from the initial point with the angle $\beta$, and if we plot from an initial point in that system the values of $\phi$ projecting the points of the spiral on the ordinates which apply to the same values of $\phi$.

As $\phi = \frac{t}{T_1}$, the time may be measured as abscissæ showing the water surface fluctuations with respect to time.

The equation 35, which determines $s$, has the same form as equation 32. It follows that for the graphical demonstration of $s$, the same method may be applied as $s$ becomes $o$ when $t/T_1 = \gamma - \beta$ and we see that the axis of the abscissæ for the rectangular system of co-ordinates which demonstrates $s$, is in the continuation of that ray

of the polar system which corresponds to the maximum and minimum values of $z$ and also of $y$. By projection of the points of the spiral to the corresponding ordinates, we get the $z$ curve, whose ordinates, as may readily be seen, cannot be measured by the same scale as those of the $z$ curve. We have to take: If (for the $z$ curve) an ordinate of one inch equals $m$ feet, then for the $s$ curve an ordinate of one inch $= \frac{m}{T}$ feet per second. The demonstration of the acceleration is similar.

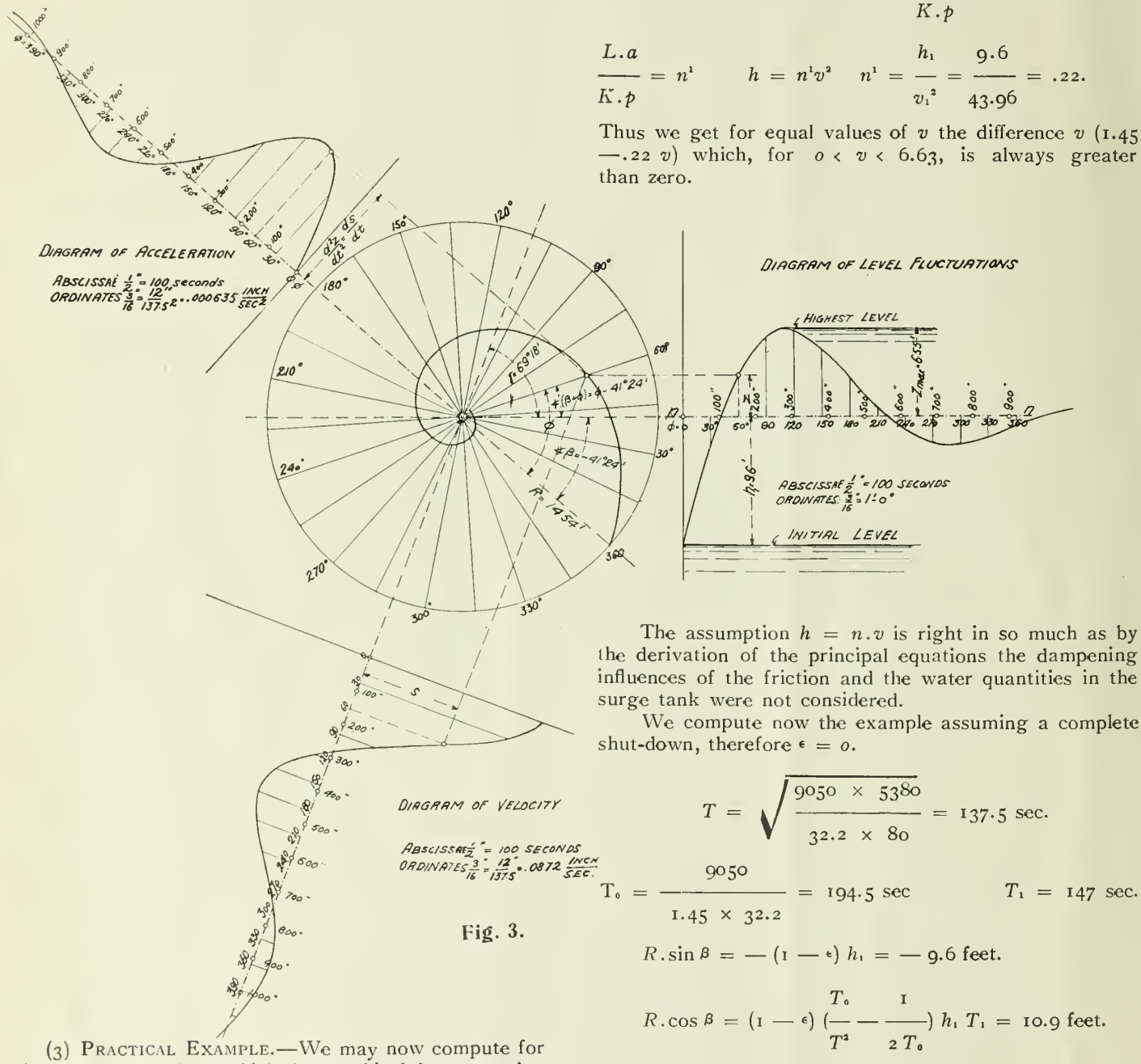
DIAGRAM OF ACCELERATION
ABSCISSÆ 1/2" = 100 seconds
ORDINATES 3/16" = 12" / 1375² .. .000635 INCH / SEC²
DIAGRAM OF LEVEL FLUCTUATIONS
HIGHEST LEVEL
INITIAL LEVEL
ABSCISSÆ 1/2" = 100 SECONDS
ORDINATES 3/16" = 1.0"
DIAGRAM OF VELOCITY
ABSCISSÆ 1/2" = 100 SECONDS
ORDINATES 3/16" = 12" / 1375 = .0872 INCH / SEC.

Fig. 3.

(3) PRACTICAL EXAMPLE.—We may now compute for this case an example to which the graphical demonstration also applies. The dimensions are the following:

$L$ = 9,050 feet $\quad p$ = 32.8 feet
$a$ = 80 square feet $\quad Q$ = 530 cubic feet per second

For a flow of 530 cubic feet per second, according to the usual method of computation, a friction coefficient $K$ = 101,000 in the formula $h = \frac{L.a}{K.p} . v^2$ and, therefore, $v_1$ = 6.63 feet per second and $h_1$ = 9.6 feet, are determined.

The surge tank section may be constant and $A$ = 5,380 square feet. According to the theory, we assume that $h = n.v$, $n = \frac{h_1}{v_1}$ = 1.45 sec. = $\frac{9.6}{6.63}$. Therefore, except for the conditions $v = o$ and $v = v_1$, the friction is greater than it would be according to the friction formula commonly used $h = \frac{L.a}{K.p} v^2$. If we take

$$\frac{L.a}{K.p} = n^1 \qquad h = n^1 v^2 \qquad n^1 = \frac{h_1}{v_1^2} = \frac{9.6}{43.96} = .22.$$

Thus we get for equal values of $v$ the difference $v$ (1.45 —.22 $v$) which, for $o < v < 6.63$, is always greater than zero.

The assumption $h = n.v$ is right in so much as by the derivation of the principal equations the dampening influences of the friction and the water quantities in the surge tank were not considered.

We compute now the example assuming a complete shut-down, therefore $\epsilon = o$.

$$T = \sqrt{\frac{9050 \times 5380}{32.2 \times 80}} = 137.5 \text{ sec.}$$

$$T_o = \frac{9050}{1.45 \times 32.2} = 194.5 \text{ sec} \qquad T_1 = 147 \text{ sec.}$$

$$R.\sin\beta = -(1 - \epsilon)\, h_1 = -9.6 \text{ feet.}$$

$$R.\cos\beta = (1 - \epsilon)\left(\frac{T_o}{T^2} - \frac{1}{2\,T_o}\right) h_1\, T_1 = 10.9 \text{ feet.}$$

$$R^2 \sin^2\beta = 92 \qquad tg\beta = -\frac{9.6}{10.9} = -.881 \qquad \beta = -41^\circ\ 24'$$

$$\frac{R^2 \cos^2\beta = 119}{R^2 = 211} \qquad \text{arc } \beta = -.722$$

$$tg\gamma = \frac{2\,T_o}{T_1} = \frac{389}{147} = 2.65 \qquad \gamma = 69^\circ\ 18'$$

$R = 14.54$ feet  arc $\gamma = 1.21$

$$z = 14.54\, e^{-\frac{t}{389}} \sin\left(\frac{t}{147} - .722\right)^*$$

$$s = .106\, e^{-\frac{t}{389}} \sin\left(1.932 - \frac{t}{147}\right).$$

The values (in seconds) for the first occurrence of the maximum and minimum are

$t_1$ max $= (\gamma - \beta)\, T_1 = [1.21 - (-.720)]\, 147 = 284$ sec.
$t_1$ min $= (\gamma - \beta + \pi)\, T_1 = (1.93 + 3.14)\, 147 = 746$ sec.

$z$ max I. $= 6.55$ feet, $z$ min II. $= -2.00$ feet,
$\delta = 2\pi T_1 = 924$ sec. $= 15$ min. 24 sec.

The graphical demonstration using these values shows very clearly the water level fluctuation in the surge tank in function of the time (Fig. 3 and Fig. 4).

(4) Work Done.—For the case of a sudden complete shut-down, it is interesting to strike a balance of the work done. The basis for such a balance may be secured by a variation of the principal equation 16, considering also the continuous equation 17. If we multiply the equation 16 by $(w.a.v\,dt)h$ we get

$$\frac{w\,a\,L}{g}\, v\,dv + (w\,a\,v.dt)\, z + (wavdt)h = o$$

$$\frac{w.a.L}{g} = M = \text{mass of the}$$

main conduit volume and for $\epsilon = o$, $w.a.v.dt$ becomes $= w\,A\,s\,dt = w.A\,dz = dG$, i.e., the increase of weight of the contents of the surge tank during the time $dt$. Also $(w.a.dt\,v)\ h = A^1 =$ to the amount of friction work in the main conduit during the time $dt$. The above-mentioned equation becomes

$$M\,\frac{dv^2}{2} + z\,dG + dA^1 = o$$

and the integration between the limits which refer to the initial condition and to the highest elevation for $z$ max., considering that at the beginning $v = v_1$ and at the end $v = o$:

$$-\frac{Mv_1^2}{2} + \int_{-h_1}^{+z\,\max} z\,dG + A^1 = o \qquad (49)$$

We may place

$$\int_{-h_1}^{+z\,\max} z\,dG = G.z_1$$

if we consider $G$ as the weight of the water that has entered the surge tank during that period and if $z_1$ is the distance of the centre of gravity of $G$ above static level $n-n$. But we may as well say $z_1 = y_1 - h_1$ (50) ($y_1$ = distance of the centre of gravity of $G$ from the initial water level). It follows then, that

$$\frac{Mv_1^2}{2} + G\,h_1 = G\,y_1 + A^1 \qquad (51)$$

This is the equation for the work done, which is correct also when the cross-section $A$ is variable. As potential energy, we must introduce: (1) The kinetic force of the volume of the main conduit, and (2) The potential energy of the weight $G$ due to the first difference of level $h_1$ between the elevation in front of the forebay, (i.e., the static level $n-n$) and the initial level.

As used work we must introduce:

(1) The lifting work due to the lift of the weight $G$ through the height $y_1$.

(2) The friction work.

We may strike the balance in the following manner:

| Useful work in foot-tons. | | Used work | |
|---|---|---|---|
| 15800 | Kinetic Energy of the Conduit Contents | | $\frac{62.5 \times 80 \times 9050}{32.2} \times \frac{6.63^2}{2}$ |
| 26600 | Potential Energy of $G = 2770$ Tons for $h_1 = 9.6$ Feet | | $5380\,(9.6+6.55) \times 62.5 \times 9.6$ |
| | Lifting Work for 2770 Tons for height of 8.07 Feet | 22400 | $\frac{16.15}{2} \cdot 2770$ |
| | Friction work | 20000 | 42,400 — 22,400 |
| 42400 | | 42400 | |

The total friction work of 20,000 foot-tons corresponds to an average friction head corresponding to the conveyed weight of 2,770 tons.

$$h_1\ (\text{average}) = \frac{20{,}000}{2{,}770} = 7.22 \text{ feet} = .752.h_1 \qquad (52)$$

We might use this result to determine the necessary surge tank volume in a similar case, if in the equation 51 we write: $A^1$ is similar to:
$h$ (average) . $G$ which must be equivalent to $.752\ h_1 G$

A supplementary formula to determine the value $h$ (average) follows in the final chapter.

*Case B.—Sudden Opening.*

Before the beginning of outflow through the penstock, the water in the main conduit and in the surge tank is quiescent. The water surface in the surge tank is at the level $n-n$ and $z$ = zero. As soon as the opening occurs, which is considered as taking place instantaneously, the outflow begins. However, the full amount of the outflow does not occur instantaneously because of the inertia of the mass; also during the time following the opening, $q$ will not be constant even with a constant opening, as with the variation of the water surface elevation the slope due to the outflow velocity also varies. As in case (*A*), where we assumed that all these influences and variations are a very small amount, we consider $q$ as constant from the instant that the opening occurs.

The following will now happen at this juncture: Immediately after the opening the water surface in the surge tank goes down. The head, due to this difference between the water level in the main conduit and that in the surge

*The proportion $t/T_1$ is in radians, but must be changed into the usual measurements as soon as we use the trigonometric functions.

tank, creates a movement in the main conduit, but on account of the inertia of the mass and the increasing friction, the velocity does not increase in the same proportion as the drop of the surface in the surge tank below the static level $n - n$. On this account the water surface in the surge tank goes below the level which corresponds to the normal water level for a constant flow. So that we have first a dropping down to a lowest level whereas afterwards there occurs a damped oscillation, until finally, under the influence of all resistances, a normal balanced condition is reached, where the level in the surge tank is as much under the level $n - n$ as the amount of head required to create the necessary slope.

The initial equation 23 has in this case (if we say $q = \epsilon Q_1$ for purposes of comparison) the same form as in case $(A)$ and therefore all other equations (growing out of equation 23) also apply. The determination of the integration constant will, however, give other values for $R$ and $\beta$ since the initial condition is different. We have

at the time $t$ = zero, $z$ = zero, and $s = s_0 = -\epsilon \frac{Q_1}{A} = -\epsilon . C_1$. For the determination of the constants $R$ and $\beta$ we get the equations:

$$R . \sin\beta = + \epsilon h_1 ; \quad R \cos\beta = -\epsilon (C_1 - \frac{h_1}{2 T_0}) T_1 \quad (53)$$

The same procedure as in case $(A)$, referring to the signs of $\sin\beta$ and $\cos\beta$, shows in this case that $\beta$ lies in the second quadrant, if $\frac{A}{a} < \frac{2L}{n^2 g}$, and in the first quadrant when $\frac{4L}{n^2 g} > \frac{A}{a} > \frac{2L}{n^2 g}$.

We get, further,

$$R = \epsilon h_1 \frac{T_1 T_0}{T^2} \qquad tg\,\beta = -\frac{1}{T_0/T_1 - \frac{1}{4} T_1/T_0} \quad (54)$$

If we open the penstock so that the outflow has the full value $q = Q_1$ ($\epsilon = 1$) then $R$ assumes the same value as in case $(A)$ for a complete shut-down. $\beta$ has in both cases the same form, therefore, (referring to the above-mentioned investigations) it follows that the initial radius vector in the polar co-ordinate system for the case $(B)$ is turned through 180° from case $(A)$. The duration of the period is the same as in case $(A)$. The graphical demonstration may be begun also from the same spiral with the slope $tg\,\alpha = -T_1/2T_0$. The analytical results (assuming a full opening from zero to 530 cubic feet per second, that is, $\epsilon = 1$, $n = 1.45$ sec., assuming the same dimensions) are as follows:

$T = 137.5$ sec. $\quad T_0 = 194.5$ sec. $\quad T_1 = 147$ sec.
$R = 14.54$ feet $\quad \beta = 180 + (-41° 24') = 138° 36'$
arc $\beta = + 2.420$

$$z = -9.6 + 14.54\, e^{-t/389} \sin(2.42 + t/147)$$

$\gamma = 69° 18'$ arc $\gamma = 1.210$

$$s = .106\, e^{-t/389} \sin(-1.21 - t/147)$$

$$= -.106\, e^{-t/389} \sin(1.932 - t/147)$$

Therefrom we determine the time for the first minimum of $z$

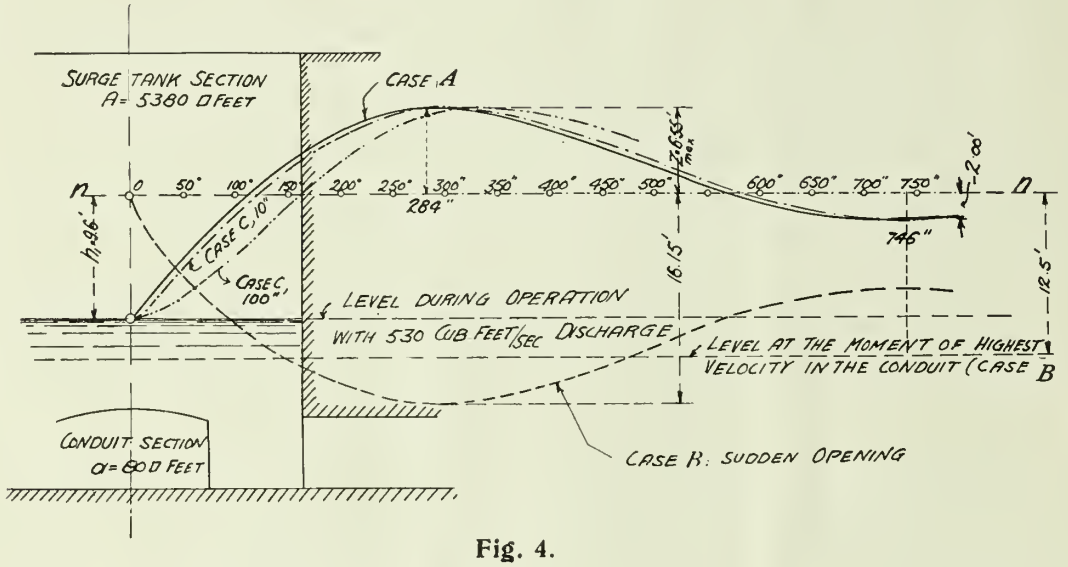


**Fig. 4.**

$$t_1 \min = 1.932 \cdot 147 = 284 \text{ sec.}$$

and for the first maximum:

$$t_1 \max = (1.932 + 3.142)\ 147 = 746 \text{ sec.}$$

These are the same values as were reached before for the occurrences of the first maximum and minimum. Finally follows

$z'_{\min} = -9.6 - 6.55 = -16.15$ feet;

$z'_{\max} = -7.6$ feet.

As $z'$ max is negative, we see that with the assumed dimensions and for full opening to 530 cubic feet-seconds, the water surface does not rise any more above the level $n - n$ after the first dip. (Fig. 4.)

The velocity in the main conduit at the time of the greatest drop becomes $v_1 = Q_1/a$ but the velocity increases continuously thereafter, as in the following period of rise of the water surface, besides the outflowing quantity $Q$, also that water from the main conduit which is necessary for the refilling of the surge tank has to flow in. And the increase lasts until $v$ becomes a maximum. Because $v.a = s\,A + Q_1$ the velocity $v$ becomes a maximum if $s$ is a maximum and we get:

$$s \max = \frac{R}{T} e^{-\frac{\pi T_1}{2 T_0}} \cdot \sin \gamma = .03 \text{ feet per second.} \qquad (55)$$

$$v \max = \frac{A}{a} \cdot s \max + v_1 = 8.65 \text{ feet per second.}$$

The conclusion of practical value is that the greatest drop below the static level $n - n$, resulting from a sudden full opening, reaches the same value as the greatest rise from the initial level for a sudden full shut-down.

*Case C: Variable Outflow.*

In the following, we consider first the influence of a gradual shut-down and opening of the penstock, and secondly the case when the outflow is variable with respect to time. The movement during the period of a gradual shut-down will be quite different from that after the shut-down. Therefore, we must consider both cases separately.

(1) Movement During the Shut-down.—We assume that the closing occurs so that the outflow decreases in direct proportion to the time which is expressed by the formula:

$$q = Q_1 \left(1 - \frac{t}{\tau}\right) \quad (56); \quad \tau = \text{time for complete shut-down}$$

Therefore,

$$c = c_1 \left(1 - \frac{t}{\tau}\right) \qquad \frac{dc}{dt} = -\frac{c_1}{\tau}$$

Similar to equation (23) we get the special principal equation (57) in the following form:

$$\frac{d^2z}{dt^2} + \frac{1}{T_0} \cdot \frac{dz}{dt} + \frac{z}{T^2} + \frac{c_1}{T_0}\left(1 - \frac{t}{\tau}\right) - \frac{c_1}{\tau} = o \qquad (57)$$

The general integral of this equation may be obtained by means of the theory of linear differential equations of the second order.

The general member is $z_1 = R\, e^{-\frac{t}{2 T_0}} \sin\left(\beta + \frac{t}{T_1}\right)$

The particular integral $z_2 = b_1 + b_2 . t$ when $b_1$ and $b_2$ may be determined by inserting the value of $z_2$, also $\frac{dz}{dt} = b_2$ and $\frac{d^2z}{dt^2} = o$ in equation 57.

Therefore,

$$\frac{b_2}{T_0} + \frac{b_1}{T^2} + \frac{b_2 t}{T^2} + \frac{c_1}{T_0} - \frac{c_1 t}{T_0 \tau} - \frac{c_1}{\tau} = o \qquad (58)$$

As the conditions of this equation must be fulfilled for all values of $t$ the two equations follow:

$$\frac{b_2}{T_0} + \frac{b_1}{T^2} + \frac{c_1}{T_0} - \frac{c_1}{\tau} = o; \qquad \frac{b_2}{T^2} - \frac{c_1}{T_0 \tau} = o \qquad (59)$$

$$\text{As } h_1 = \frac{c_1 T^2}{T_0} \qquad b_1 = h_1 \left[\frac{T_0}{\tau}\left(1 - \frac{T^2}{T_0^2}\right) - 1\right] \qquad b_2 = \frac{h_1}{\tau}$$

The general integral is obtained by the form $z = z_1 + z_2$. Therefore,

$$z = R . e^{-t/2T_0} \sin(\beta + t/T_1) + b_1 + b_2 t \qquad (60)$$

And with differentiation with respect to $t$ $\left(tg\ \gamma = \frac{2 T_0}{T_1}\right)$

$$s = \frac{R}{T} e^{-t/2T_0} \sin(\gamma - \beta - t/T_1) + b_2 \qquad (61)$$

In order to obtain the constant for the integration, consider that for $t = o$, $z = z_0 = -h_1$, as in case $(A)$, but that $s = s_0 = o$, because the shut-down occurs gradually. Therefore:

$$R \sin\beta = -h_1 \frac{T_0}{\tau}\left[1 - \frac{T^2}{T_0^2}\right]$$

$$R \cos\beta = -h_1 \frac{T_1}{2\tau}\left[3 - \frac{T^2}{T_0^2}\right]$$

$$R = h_1 \cdot \frac{T_0}{\tau} \cdot \frac{T_1}{T} \qquad (62) \qquad tg\,\beta = 2 \frac{1 - \frac{T^2}{T_0^2}}{3 - \frac{T^2}{T_0^2}} \cdot \frac{T_0}{T_1} \qquad (63)$$

For the graphical demonstration of these functions we may separate them into three equations. For instance, the $z$ — function into

$$r = R\, e^{-t/2T_0 \phi}; \quad z_1 = r \sin(\beta + \phi); \quad z_2 = b_1 + (b_2 T_1)\phi. \qquad (64)$$

The value of $r$ determines again a logarithmic spiral with the slope $tg\,\alpha = -\frac{T_1}{2 T_0}$ with the initial vectors $r_0$ and $\beta$.

$z_1$ is obtained in the rectangular co-ordinate system by projection from the polar system as in the former cases. $z_2$ in the same rectangular co-ordinate system is a straight line which intersects the ordinate axis at the distance $b_1$ from the initial point and whose inclination to the axis of abscissæ is fixed by the direction constant $b_2 . T_1$. The algebraic sum of $z_1 + z_2$ gives $z$. (See Fig. 5.) But only that part of the curve which lies between the values of the abscissæ $o$ and $\tau$ is of practical importance because at the time $\tau$ the complete shut-down has already occurred (as we assumed).

(2) Movement After Completed Shut-down.—After the shut-down has occurred, the movement of the water surface continues as described in case $(A)$. Therefore, from this moment the following equations are effective:

$$z = R_1 e^{-t/2T_0} \sin(\beta_1 + t/T_1) \qquad (65)$$

$$s = \frac{R_1}{T} e^{-t/2T_0} \sin(\gamma - \beta_1 - t/T_1) \qquad (66)$$

(see equations 32 and 35).

Where $tg\ \gamma = \frac{2 T_0}{T_1}$

The constants $R_1$ and $\beta_1$ have to be determined from the value of $z$ and $s$ computed for $t = \tau$. In order to obtain an easier computation, we measure the time for

the second period from the moment of the completed shut-down, so that the following equations for the determination of the constants $R_1$ and $\beta_1$ are in effect:

$$R_1 \sin \beta_1 = z_T$$

$$R_1 \cos \beta_1 = (s_T + \frac{z_T}{2 T_0})$$

where $z_T$ and $s_T$ are limiting values of the first period.

(3) Practical Example.—The computation in connection with the preceding example follows for a shut-down in 10, 100 and 200 seconds, with a discharge of 530 cubic feet per second.

From the graphical demonstration (Fig. 4) we see that the period of shut-down up to 100 seconds influences the maximum elevation a very small amount indeed. The explanation for this is that during this period the velocity

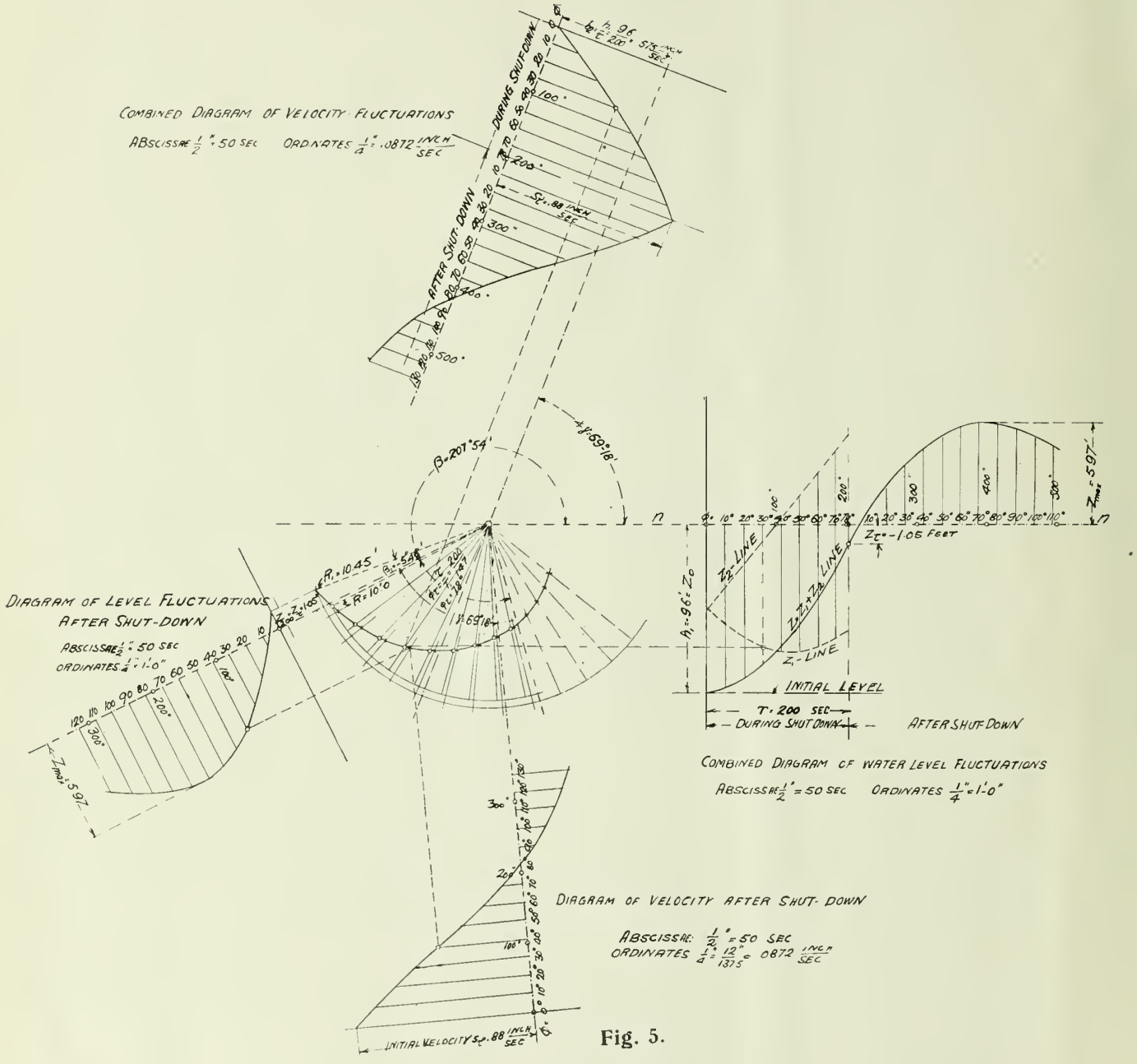


Fig. 5.

The demonstration of the movement may be developed according to the same method as in case (*A*). For the computation of the value of the most practical importance (the maximum elevation), we get

$$z \max = R_1 \, e^{-\frac{T_1}{2 T_0}(\gamma - \beta_1)} \sin \gamma \qquad (67)$$

of flow in the main conduit decreases very little. For the determination of the dimensions of the surge tank, we have, therefore, to consider the results of the limiting case, that is, the sudden shut-down. The computation for that condition is a much simpler one, (case *A*). For a gradual opening, the same relations exist. The lowering of the level $n-n$ occurs to the same extent and for the same duration as the rise above the initial level in the case already computed.

With $T$ = 137.5 seconds, $T_0$ = 194.5 seconds, $T_1$ = 147 seconds, $h_1$ = 9.6 feet, we get the following table:

| for $\tau$ = | 10 sec. | 100 sec. | 200 sec. |
|---|---|---|---|
| $b_1 = h_1 \left[\frac{T_0}{\tau}\left(1-\frac{T^2}{T_0^2}\right)-1\right] =$ | +83.3 feet | —.264 feet | — 4.93 feet |
| $b_2 = \frac{h_1}{\tau} =$ | .96 feet/sec. | .096 feet/sec. | .048 feet/sec. |
| $R = h_1 \cdot \frac{T_0}{\tau} \cdot \frac{T_1}{T} =$ | 200 feet | 20 feet | 10 feet |
| $tg\beta = \frac{2\left(1-\frac{T^2}{T_0^2}\right)T_0}{\left(3-\frac{T^2}{T_0^2}\right)T_1} =$ | ·530 | ·530 | ·530 |
| | $\beta = 207° 54'$ | arc $\beta = 3.628$ | $\gamma = 69° 18'$; arc $\gamma = 1.210$ |
| $z_\tau$ = | —9.10 feet | —4.85 feet | —1.05 feet |
| $s_\tau$ = | .097 feet/sec. | .091 feet/sec. | .0735 feet/sec. |
| $R_1 \sin \beta_1$ = | —9.10 feet | —4.85 feet | —1.05 feet |
| $R_1 \cos \beta_1$ = | +11.05 feet | +11.6 feet | +10.4 feet |
| $R_1$ = | 14.30 feet | 12.55 feet | 10.45 feet |
| $\beta_1$ negative = | —39° 30′ | —22° 44′ | —5° 46′ |
| arc $\beta_1$ = | —.6894 | —.3968 | —.1006 |
| $z$ max = | +6.53 feet | +6.40 feet | +5.97 feet |

Of special interest is the case of an outflow, which is variable in the sense that the outflow increases considerably during a certain time and decreases afterwards to the same amount as before or to some other amount, (for instance, in a plant for railway operation).

1. Analytical Investigation.—We assume now, that under circumstances similar to those above mentioned, the following law of outflow is effective:

$$q = \epsilon . Q_1 (1 + f \sin t/\tau)$$

so that for the time

$$t = o: \quad t = \frac{\pi . \tau}{2}; \quad t = \pi\tau \quad q \text{ becomes resp. } = \epsilon Q_1; \quad \epsilon Q_1 (1+f); \quad \epsilon Q_1 \quad - \quad - \quad (68)$$

after the time $t = \pi . \tau$ the outflow may be constant again.

$\epsilon$ and $f$ are natural numbers. $f$ is the proportion of the maximum increase of the outflow to the normal outflow. So that

$$c = \frac{q}{A} = \epsilon c_1 \left(1 + f \sin \frac{t}{\tau}\right) \qquad \frac{dc}{dt} = \epsilon . f . \frac{c_1}{\tau} \cos \frac{t}{\tau}$$

Equation 23 may then be written:

$$\frac{d^2z}{dt^2} + \frac{1}{T_0}\cdot\frac{dz}{dt} + \frac{z}{T^2} + \frac{\epsilon . c_1}{T_0} + \frac{\epsilon . f . c_1}{T_0} \sin \frac{t}{\tau} + \frac{\epsilon . f . c_1}{\tau} \cos \frac{t}{\tau} = o \quad (69)$$

If we introduce

$$z = y - \frac{\epsilon c_1 T^2}{T_0} = y - \epsilon h_1; \quad \frac{dz}{dt} = \frac{dy}{dt}; \quad \frac{d^2z}{dt^2} = \frac{d^2y}{dt^2} \quad (70)$$

and further,

$$\epsilon . f . c_1 \left[\frac{1}{T_0} \sin \frac{t}{\tau} + \frac{1}{\tau} \cos \frac{t}{\tau}\right] = \epsilon . f . c \sqrt{\frac{1}{T_0^2} + \frac{1}{\tau^2}} \sin \left(\phi + \frac{t}{\tau}\right) \quad (71)$$

with $tg\,\phi = \frac{T_0}{\tau}$ then the equation 69 transposes to

$$\frac{d^2y}{dt^2} + \frac{1}{T_0}\frac{dy}{dt} + \frac{y}{T^2} + \epsilon . f . c_1 \sqrt{\frac{1}{T_0^2} + \frac{1}{\tau^2}} \sin \left(\phi + \frac{t}{\tau}\right) = o \quad (72)$$

and it then follows again from the theory of differential equations of the second order that $y = y_1 + y_2$, that is,

$$y_1 = R . e^{-t/2T_0} \sin (\beta + t/T_1) \text{ general integral} \quad (73)$$

$$y_2 = b \sin \left(\psi + \frac{t}{\tau}\right) \text{ particular integral} \quad (73)$$

The latter may also be written

$$y_2 = b . \sin \psi \cos \frac{t}{\tau} + b \cos \psi \sin \frac{t}{\tau} \quad (74)$$

The integration constants $R$ and $\beta$ may be obtained from the initial phase and the constants $b$ and $\psi$ by forming the equations

$$\frac{dy_2}{dt} = \frac{b}{T}\cos\left(\psi + \frac{t}{T}\right) = \frac{b}{T}\cos\psi\cos\frac{t}{T} - \frac{b}{T}\sin\psi\sin\frac{t}{T}$$

$$\frac{d^2y_2}{dt^2} = -\frac{b}{T^2}\sin\left(\psi+\frac{t}{T}\right) = -\frac{b}{T^2}\sin\psi\cos\frac{t}{T} - \frac{b}{T^2}\cos\psi\sin\frac{t}{T}$$

We then introduce these values in the differential equation and combine the members containing cos $t/\tau$ and sin $t/\tau$

$$\left[-\frac{b}{T^2}\sin\psi + \frac{b}{\tau\,T_0}\cos\psi + \frac{b}{T^2}\sin\psi + \right.$$

$$\left. \epsilon.f.c_1\sqrt{\frac{1}{T_0^2}+\frac{1}{T^2}}.\sin\phi\right]\cos\frac{t}{\tau} +$$

$$\left[-\frac{b}{T^2}\cos\psi - \frac{b}{\tau\,T_0}\sin\psi + \frac{b}{T^2}\cos\psi + \right.$$

$$\left. \epsilon.f.c_1\sqrt{\frac{1}{T_0^2}+\frac{1}{T^2}}.\cos\phi\right]\sin\frac{t}{\tau} = o$$

and as these equations hold good for all values of $t$, the terms in both parentheses must become $o$. One thus obtains two equations with the unknowns $b$ and $\psi$ and considering that $tg\,\phi = \frac{T_0}{\tau}$ and $h_1 = \frac{c_1 T^2}{T_0}$, it follows

$$b = -\epsilon.f.h_1\frac{\sqrt{1+\frac{T_0^2}{T^2}\left[\frac{T^2}{T^2}+\frac{T^2}{T_0^2}-1\right]^2}}{1+\frac{T^2}{T^2}\left[\frac{T^2}{T^2}+\frac{T^2}{T_0^2}-2\right]}$$

$$tg\,\psi = -\frac{T_0}{\tau}\cdot\left[\frac{T^2}{T^2}+\frac{T^2}{T_0^2}-1\right] \qquad (75)$$

For the determination of the integration constants $R$ and $\beta$, it must be considered that for

$$t = o; \quad z = -\epsilon h_1; \quad y = o; \quad s = o.$$

We obtain for this case the equations

$$\begin{aligned} R.\sin\beta &= -b.\sin\psi \\ R.\cos\beta &= -b\left[\frac{T_1}{\tau}\cos\psi + \frac{T_1}{2\,T_0}\sin\psi\right] \end{aligned} \qquad (76)$$

During the variation of the outflow the following equations are effective:

$$\left.\begin{aligned} z &= \epsilon.h_1 + R\,e^{-\frac{t}{2\,T_0}}\sin\left(\beta+\frac{t}{T_1}\right) + b\sin\left(\psi+\frac{t}{\tau}\right) \\ s &= \frac{R}{T}e^{-\frac{t}{2\,T_0}}\sin\left(\gamma-\beta-\frac{t}{T_1}\right) + \frac{b}{\tau}\cos\left(\psi+\frac{t}{\tau}\right) \end{aligned}\right\} \text{ with } tg\,\gamma = \frac{2\,T_0}{T_1}$$

$\psi$ amounts generally to about $\frac{3\pi}{2}$ which relation simplifies the computation of $R$ and $\beta$.

The variation ceases (as we assumed) after the time $t = \pi.\tau$. The values which correspond to the elevation of the water surface and velocity at that time are obtained by the formulæ:

$$\left.\begin{aligned} z_\tau &= -\epsilon h_1 + R\,e^{-\frac{\tau}{2\,T_0}}\sin\left(\beta+\pi\frac{\tau}{T_1}\right) - b.\sin\psi \\ s_\tau &= \frac{R}{T}e^{-\frac{\tau}{2\,T_0}}\sin\left(\gamma-\beta-\pi\frac{\tau}{T_1}\right) - \frac{b}{\tau}\cos\psi \end{aligned}\right\} \qquad (77)$$

For further investigation, the formulæ of case $(A)$ may be applied. In order to simplify matters, in the determination of the further movement, we measure the time from the instant of the beginning of the constant outflow and logically we must use the limiting values of the preceding phase for the determination of the integration constants $R_1$ and $\beta_1$.

If we do not hinder the variation of the outflow but maintain the law $\epsilon Q_1\,(1 + f\sin\frac{t}{\tau})$, we see that the movement of the water surface in the surge tank takes the form of a forced oscillation; where the influence of the first member decreases with the increase of $t$ and this the quicker the larger the value of $\frac{T_1}{2\,T_0}$ becomes in the member

$$e^{-\frac{t}{2\,T_0}} = e^{-\frac{T_1}{2\,T_0}\cdot\frac{t}{T_1}}$$

The movement of the level of the water surface is merely that of a harmonic oscillation. In such cases it is well known that the phenomenon of resonance may occur, if the period of the actuating influence has the same duration as the swinging bodies' own period, that is to say, if, in the case mentioned $\tau = T$. The value of the amplitude of the forced oscillation is then

$$b = -\epsilon f h_1\frac{\sqrt{1+\frac{T^2}{T_0^2}}}{T^2/T_0^2} = -\epsilon f h_1\sqrt{\frac{T_0^2}{T^2}+1}\cdot\frac{T_0}{T} \qquad (78)$$

We obtain the angle $\psi$ from $tg\,\psi = -\frac{T}{T_0} = -\frac{1}{tg\,\phi}$ (79)

because $\frac{T_0}{\tau} = tg\,\phi$ and $\tau = T$.

According to the theory of the phenomena of resonance, the amplitude of the forced oscillation becomes of infinite value when the damping force is infinitely small. The latter would be the case, using our terms, when $T_0 =$ infinite with which value the amplitude of the above-mentioned equation also becomes infinite. The phase difference between force and movement has thereby to be

equal to $\frac{\pi}{2}$, which also finds its expression in this equation (79).

2. Graphical Demonstration.—The graphical demonstration of the movement of the first phase is not difficult. The demonstration of the values

$$z_1 = R\, e^{-\frac{t}{2\,T_0}} \sin\left(\beta + \frac{t}{T_1}\right)$$

may be carried out in the same form as before by means of the projection of the logarithmic spiral. The demonstration of

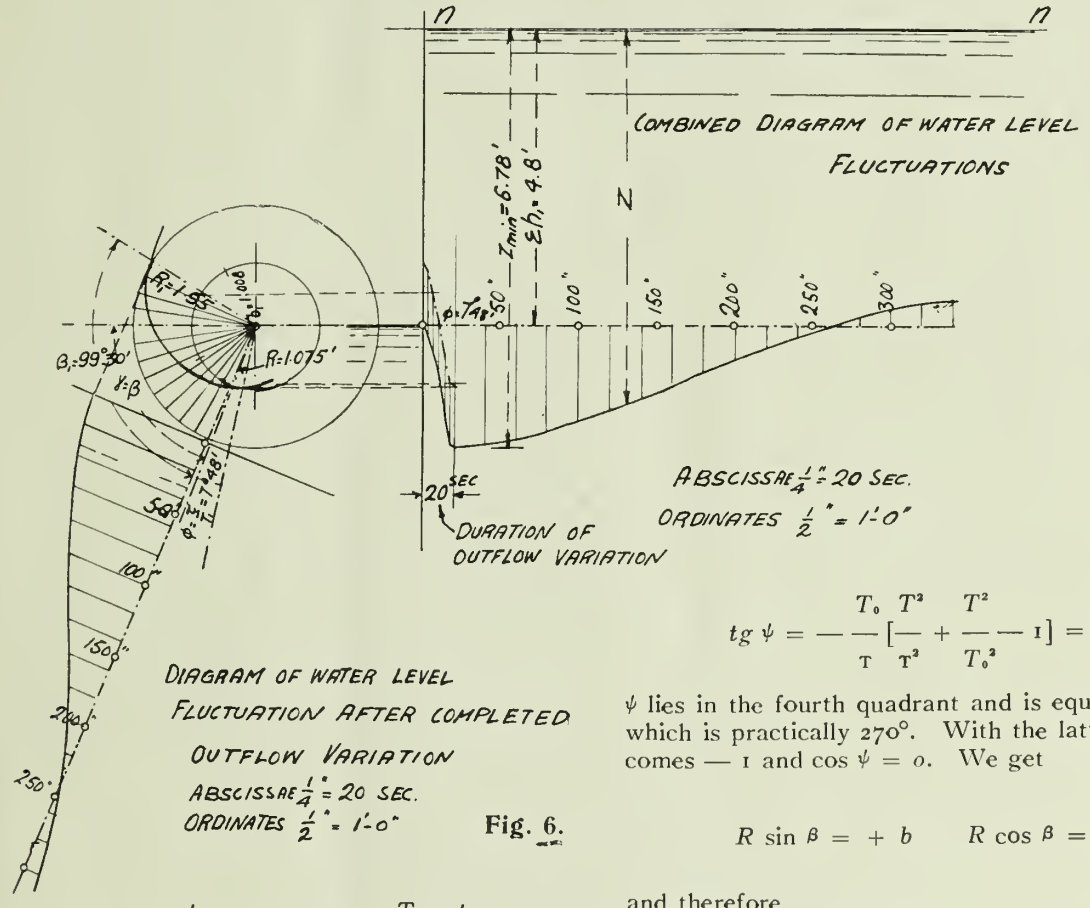


Fig. 6.

$$z_2 = b \cdot \sin\left(\psi + \frac{t}{T}\right) = b \sin\left(\psi + \frac{T_1}{T} \cdot \frac{t}{T_1}\right)$$

is obtained by projection of a circle, constructed with the radius $b$ in the polar system, where the initial vector is inclined against the axis of abscissæ with an angle $\psi$.

In this case a rotation of the initial vector of $z_1$ by an amount of $2\pi$ corresponds to a rotation of the initial vector of $z_2$ by an amount equal to $\frac{T_1}{T} 2\pi$. (See Fig. 6.)

3. Analytical Example.—We use the same conduit and surge tank dimensions and assume the same friction conditions. As before, we take $\epsilon = 0.5$; with $f = 1$ and for $T = 20$ seconds we have an initial flow of 265 cubic feet per second which increases during the time $\frac{\pi}{2} \cdot T = 31.4$ seconds to 530 cubic feet per second, and decreases after another period of 31.4 seconds (a total of 62.8 seconds) to 265 cubic feet per second. Therefore, with $T = 137.5$ sec. $T_0 = 194.5$ sec. $T_1 = 147$ sec. $h_1 = 9.6$ ft.

we get $\frac{T_0}{T} = 9.725$; $\frac{T}{T} = 6.875$; $\frac{T}{T_0} = .707$

$$b = -\epsilon f h_1 \cdot \frac{\sqrt{1 + \frac{T_0^2}{T^2}\left[\frac{T^2}{T^2} + \frac{T^2}{T_0^2} - 1\right]^2}}{1 + \frac{T^2}{T^2}\left[\frac{T^2}{T^2} + \frac{T^2}{T_0^2} - 2\right]} = -1.008 \text{ feet}$$

$$tg\,\psi = -\frac{T_0}{T}\left[\frac{T^2}{T^2} + \frac{T^2}{T_0^2} - 1\right] = -457.1$$

$\psi$ lies in the fourth quadrant and is equal to 270° 7′ 50″, which is practically 270°. With the latter value $\sin\psi$ becomes $-1$ and $\cos\psi = 0$. We get

$$R \sin\beta = +b \qquad R\cos\beta = +\frac{T_1}{2\,T_0}$$

and therefore

$$tg\,\beta = \frac{2\,T_0}{T_1} = tg\,\gamma \qquad \beta = \gamma = 69°\ 18'$$

$$R = b\sqrt{1 + \frac{T_1^2}{(2\,T_0)^2}} = b \cdot T_1 \sqrt{\frac{1}{T_1^2} + \frac{1}{(2\,T_0)^2}} =$$

$$b \cdot \frac{T_1}{T} = -1.008 \cdot \frac{147}{137.5}$$

$R = -1.075$ feet, therefore

$$z = -4.8 - 1{,}075 \cdot e^{-\frac{t}{389}} \sin\left[69°\ 18' + \frac{t}{147}\right] + 1.008 \cos\frac{t}{20}$$

$$v = + .00782\, e^{-\frac{t}{389}} \sin \frac{t}{147} - .0503 \sin \frac{t}{20}$$

If at the time $t = \pi . T$ the fluctuation of the outflow ceases it follows that

$$z_T = -4.8 - 1.075\, e^{-\frac{20}{389}\pi} \sin\left(69^\circ 18' + \frac{20}{147}\right) - 1.01 = 2.88 \text{ ft.}$$

$$v_T = .00782\, e^{-\frac{20}{389}\pi} \sin \frac{20}{147} = .00275 \text{ feet per second.}$$

For the movement that follows, we may use the equations (32) and (35)

$$z = -\epsilon h_1 + R_1\, e^{-\frac{t}{2T_0}} \sin\left(\beta_1 + \frac{t}{T_1}\right)$$

$$s = \frac{R_1}{T} e^{-\frac{t}{2T_0}} \sin\left(\gamma - \beta_1 - \frac{t}{T_1}\right) \quad \text{with } tg\,\gamma = \frac{2T_0}{T_1}$$

and $R_1$ and $\beta_1$ may be determined by the conditions $t = o$, $z = z_T$, $s = s_T$. We obtain

$R_1 . \sin \beta_1 = -9.6 + 2.88 + 4.8 = -1.92$ feet

$R_1 . \cos \beta_1 = .00275 \cdot 147 - 1.92 \frac{147}{389} = -.321$ feet

$R_1 = 1.95$ feet; $\beta_1$ lies in the third quadrant, i.e., negative.

$tg\,\beta_1 = 5.98975$ $\quad \beta_1 = -99^\circ 30'$ $\quad$ Arc $\beta_1 = -1.737$

$\gamma = 68^\circ 18'$ $\quad$ arc $\gamma = 1{,}210$

$$z = -4.8 + 1.95\, e^{-\frac{t}{389}} . \sin\left[-99^\circ 30' + \frac{t}{147}\right]$$

$$s = -.0142\, e^{-\frac{t}{389}} . \sin\left[11^\circ 12' + \frac{t}{147}\right]$$

Since at the time $t = o$ the velocity $s$ is positive, a maximum first occurs. The time from the beginning of that maximum until the beginning of the second phase is $t_1 = 147\,(\gamma - \beta_1) = 433$ seconds.

$$z_1 \min = -4.8 + 1.95\, e^{-\frac{433}{389}} \sin\gamma = -4.19 \text{ feet}$$

The general course of the movement may be seen in Fig. 6.

In case of a continuous fluctuation of flow, expressed by the law $q = \epsilon Q_1 \left(1 + f \sin \frac{t}{T}\right)$ with $\epsilon = .5$ and $f = 1$ the forced oscillation becomes nearly a constant harmonic oscillation of the amplitude one foot measured from $z = -4.8$ feet and for a duration of period of 137.6 seconds.

The case of resonance might occur if $T = T = 137.5$ seconds. This would be the case for an amplitude of the forced oscillation (measured from $z = -4.8$ feet) equal to

$$b = \pm \epsilon . f . h_1 \sqrt{\frac{T_0^2}{T^2} + 1} . \frac{T_0}{T} = \pm 11.7 \text{ feet.}$$

Herein the duration of period would be $T . 2\pi = 137.5 . 2 . \pi$ sec $= 863$ $= 14' 23''$. It does not seem impossible for such periods to occur in regular operations as for railways, so that this investigation should be useful.

The preceding method, introducing a periodical function for the determination of a variable outflow in function of the time may be enlarged, introducing

$$q = \epsilon Q_1 \left[f_0 + f_1 \sin\left(\phi_1 + \frac{t}{T}\right) + f_2 \sin\left(\phi_2 + 2\frac{t}{T}\right) + \ldots\right] \quad (80)$$

where we find the values for $f$ and $\phi$ corresponding to a given variation of $q$ by means of the Fourier series. The integration of the differential equation thus obtained involves no difficulty whatever. It is based upon the same method as that given in the last example. The values of $z$ and $s$ have the form

$$z = z_0 + R . e^{-\frac{t}{2T_0}} \sin\left(\beta + \frac{t}{T_1}\right) + \Sigma k_n \sin\left(\psi_n + n\frac{t}{T}\right)$$

$$s = s_0 + \frac{R}{T} e^{-\frac{t}{2T_0}} \sin\left(\gamma - \beta + \frac{t}{T_1}\right) + \Sigma \frac{n . s_n}{T} \cos\left(\psi_n + n\frac{t}{T}\right) \quad (81)$$

Naturally the computation requires great care. The graphical demonstration is obtained by superposition of the projections obtained from the polar system with the

$$-\frac{T_1}{2T_0} \cdot \frac{t}{T_1}$$

logarithmic spirals $r = R\, e$ and the circles with the radii $k$.

*Case D: Involving Spillways.*

In order to decrease the surge in the surge tank, a spillway may be introduced in the surge tank or in the main conduit. The problem of interest, then, is to determine at what elevation and with what capacity such a spillway should be designed in order that the surge shall not exceed a certain elevation. The case will be investigated under the following assumptions: The spillway is attached to the surge tank (Fig. 7). The crest of the spillway is at the distance $e'$ from the static level $n-n$; $e'$ is positive if the crest is above $n-n$ and negative if below. The width of the spillway is $b'$. The outflow of $Q$ cubic feet per second is suddenly interrupted.

We must take care of several periods of movement. Following a complete shut-down, the water surface in the surge tank will rise to the height of the spillway crest, as is described in case (*A*). Next, the rise will continue, but

water will flow over the spillway up to a maximum elevation, then decrease to the elevation of the spillway crest. At this moment, the outflow ceases. The movement then takes the form of the case (*A*), and retains this form thereafter, provided the crest of the spillway is not reached again. Otherwise, the same cycle is repeated or the con-

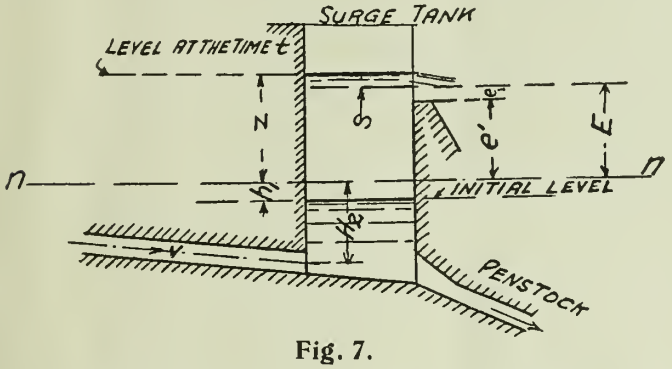


**Fig. 7.**

dition remains one of constant overflow. Therefore, for the first phase the following equations are effective:

$$z = R.e^{-\frac{t}{2T_0}} \sin\left(\beta + \frac{t}{T_1}\right)$$

$$s = \frac{R}{T} e^{-\frac{t}{2T_0}} \sin\left(\gamma - \beta - \frac{t}{T_1}\right) \quad \text{with } tg\,\gamma = \frac{2T_0}{T_1}$$

The integration constants $R$ and $\beta$ are obtained for the conditions $t = o$, $z = -h_1$, $s = c_1$. The final value of $z$ is in this period of movement $e'$. From this we determine the time $t_e'$, which is necessary for the rise up to the level of the spillway crest and by means of the second equation we determine the final velocity $s_e'$. In the second period of movement now beginning, we have overflow on the spillway, that is, according to the familiar formula for spillways $q = \frac{2}{3}\mu\, b'\, h'\, \sqrt{2gh'}$ where $h' = z - e'$ represents the overflow height and we get

$$\frac{q}{A} = c = \frac{2}{3}\mu\, b' \frac{\sqrt{2gh'^3}^2}{A} \qquad (82)$$

The introduction of this formula and its derivation in the principal equation (23) would lead to a differential equation higher than the first degree, the integration of which might be accomplished by development in series. For practical purposes, however, an easier but sufficiently exact approximation may be obtained in the following manner:

If we plot the different values of $h'$ computed for the width $b'$ on a rectangular co-ordinate system, the values $h'$ as abscissæ, and the values of $q$ as ordinates (see Fig. 8), we obtain a parabolic curve, passing through the origin. The velocity $s_e'$ (which is the final value of the velocity of the surface at the end of the first period of movement) together with $A$ determines a quantity of water $s_e'.A$, which is certainly larger than the maximum value of the overflowing quantity during the second period. If we now draw tangents from the point of the quantity curve, which corresponds to $A\,s_e'$ and assume as a preliminary approximation that within the heights of the overflow, which corresponds to the intersection of the tangent with the axis of the abscissæ, the overflowing quantity equals zero, which is the same as if we move the elevation of the spillway crest higher up by the same amount, and if we take from this location the water quantity proportional to the height of this new elevation of the spillway crest corresponding to the tangent, then we get a linear equation for the determination of the water quantity, whereas the differential equation for the determination of the elevations also becomes linear.

The results thus obtained from these equations are naturally only approximate ones, as the overflowing quantities are introduced as too small. The computed values of $z$ exceed the actual values. For practical purposes, this first approximation is generally sufficient, but we have no difficulties using the results of the first approximation for a second computation, drawing the tangent on that point of the overflow curve which corresponds to the maximum value of the elevation found in the first computation and repeating with these results the computation as before. We may use this second approximation in the first computation if instead of the value for the point in the curve, we take a somewhat smaller value, say, $u\,A\,s_e'$, when $u = 0.7$ to $0.8$. We use the latter method in the following:

The overflow height, which gives an overflow of $u.A.s_e'$, is determined by

$$h_u = \left(\frac{3}{2}\frac{u}{\mu} \cdot \frac{s_e' A}{b'\sqrt{2g}}\right)^{2/3}$$

The proportional factor $k$ for the linear variation of $q$ is obtained by differentiation of $q$ with respect to $k$. Therefore:

$$k = \frac{dq}{dh'} = (\text{as } h' = h_u) = \mu\, b' \sqrt{2g\, h_u} = \sqrt[3]{3/2\ \mu^2\, b'^2\, 2g\, u\, s_e' A}$$

$k$ has the dimension $l^2.t^{-1}$, and the value of the abscissa $e_1$, which is the difference between the true elevation of the spillway crest and that obtained by ap-

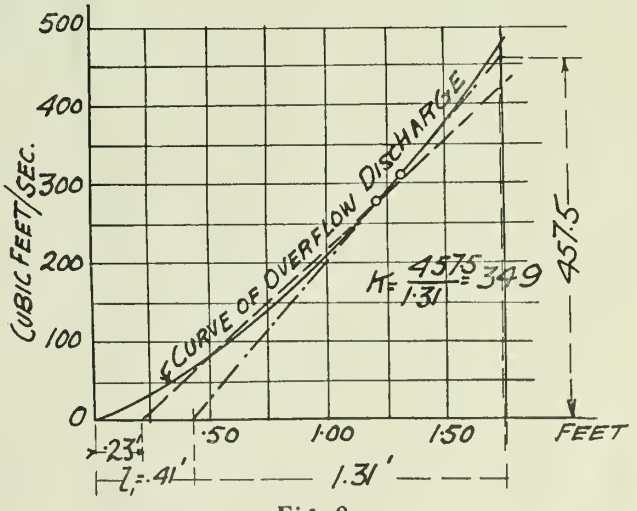


**Fig. 8.**

proximation, is

$$e_1 = h_u - \frac{u.s_e'.A}{k} \qquad e_1 = \sqrt[3]{\frac{19}{12}}\left(\frac{u\, s_e' A}{\mu\, b'\sqrt{2g}}\right)^{2/3} \qquad (84)$$

The values are easiest obtained graphically from the curve of the overflow quantities.

Therefore, with $e_1$ the height of the ideal spillway crest above the static level $n - n$, (that is $E = e' + e_1$)

once determined, the computation of the first period of movement must be extended to the elevation $E$.

We obtain from the previously mentioned formulæ $z_e = E$ and $s_e$. These are initial values for the second phase, from which beginning we measure the time anew.

$$c = \frac{q}{A} = \frac{k}{A}(z - E) \qquad \text{and therefore} \quad \frac{dc}{dt} = \frac{k}{A}\frac{dz}{dt}$$

and the equation 23 becomes

$$\frac{d^2z}{dt^2} + \left(\frac{1}{T_0} + \frac{k}{A}\right)\frac{dz}{dt} + \left(\frac{1}{T^2} + \frac{k}{A\,T_0}\right) z - E.\frac{k}{A.T_0} = o$$

Introducing $y = z + m = z - \dfrac{E}{\dfrac{A\,T_0}{k\,T^2} + 1}$ and abbreviating

$$\frac{1}{T_0} + \frac{k}{A} = \frac{1}{T_0^1}; \quad \frac{1}{T^2} + \frac{k}{A\,T_0} = \frac{1}{(T^1)^2} \text{ we get}$$

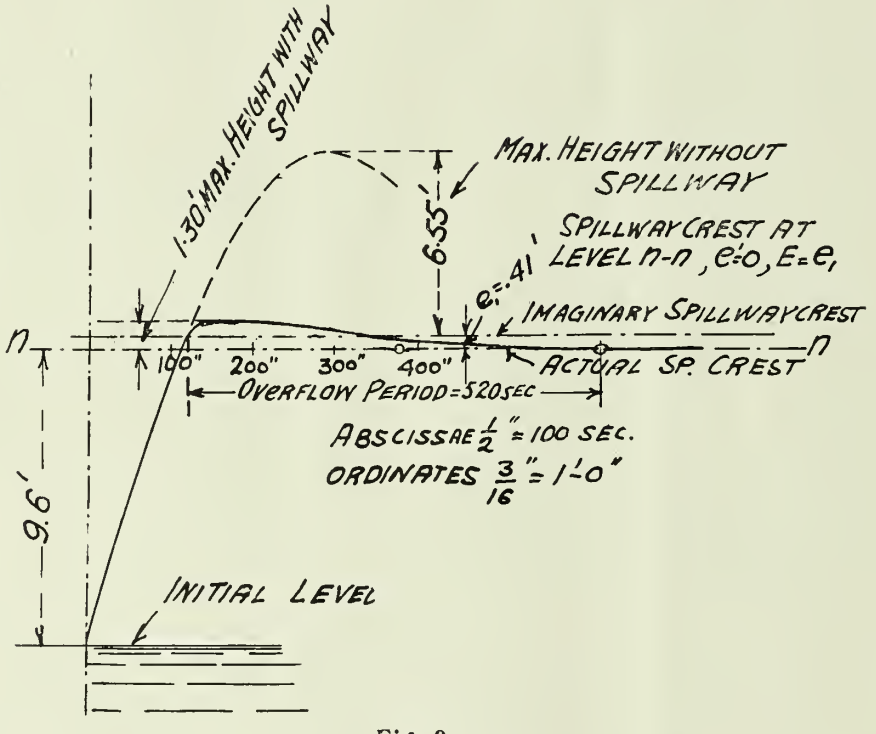


Fig. 9.

$$\frac{d^2y}{dt^2} + \frac{1}{T_0^1}\frac{dy}{dt} + \frac{y}{(T^1)^2} = o \qquad 85)$$

Corresponding to the investigations regarding the form of the general integral of this differential equation, we must investigate whether the difference

$$\frac{1}{(T_1^1)^2} = \frac{1}{(T^1)^2} - \frac{1}{(2\,T_0^1)^2} \text{ is positive or zero, or negative,}$$

which we obtain by substituting the values of $\dfrac{1}{T_0^1}$ and $\dfrac{1}{(T^1)^2}$

$$\frac{1}{(T_1^1)^2} = \frac{1}{T_1^2} + \frac{k}{2\,A}\left(\frac{1}{T_0} - \frac{k}{2\,A}\right)$$

by which formula the investigation mentioned may be carried out and the corresponding form of the general integral may be used.

The integration constants must be determined with the initial values

$$t = o; \quad z_0 = E; \quad s_0 = s_e$$

The duration of the second period of movement is obtained from the equation for $z$, which is given by that value of $t$ for which $z$ becomes $E$ once more. If that does not occur in a case of non-periodic movement, for instance, if the spillway crest lies below the level $n-n$, then the duration of the second period of movement is only limited by a new occurrence of any kind of outflow. Otherwise, the final values of the second period are the initial values of a following period, which must be handled the same as the first case. (Case $A$.)

The method of computation may be shown best by an example. Using the former example, we consider a spillway of 65.7 feet width, the crest of which is at the static level $n-n$. That is, for this assumption $e' = $ zero.

The flow of 530 cubic feet per second is suddenly stopped.

From the results of case ($A$) we get

$$z_e' = o; \qquad t_e' = 106 \text{ sec.}; \qquad s_e' = +\ .075 \text{ feet/sec.}$$

The velocity $s_e'$ corresponds to the flow in the surge tank at the time $t_e'$ of $q_e' = .075 \cdot 5380 = 404 \dfrac{\text{cu.ft.}}{\text{sec.}}$. For a spillway width of 65.7 feet and for $\mu = 0.6$, we get from the spillway formula $q \dfrac{\text{cu. ft.}}{\text{sec.}} = 208\ h'\ \sqrt{h'}$ and therefore for

$$u \cdot q_e' = 282 \frac{\text{cu. ft.}}{\text{sec.}} \quad (u = .7)$$

an overfall height of $h' = 1.22$ feet and a proportional factor $k = 3/2 \cdot 2.08 \cdot h_u^{\frac{1}{2}} = 345 \frac{\text{sq. ft.}}{\text{sec.}}$ and therefore as the distance of the ideal spillway crest from the static level $n-n$ because $e' =$ zero; $E = .410$ and with the results of case a for $z = E$

$$s_0 = + .073 \text{ feet per second}$$

In order to determine which integral formula to use, we have

$$\frac{1}{(T_1')^2} = \frac{1}{T_1^2} + \frac{k}{2A}\left(\frac{1}{T_0} - \frac{k}{2A}\right) = -\frac{1}{34.6^2}$$

$$\left(\text{therefore } \frac{1}{T_2^2} = -\frac{1}{(T')^2} = \frac{1}{(2T_0')^2} - \frac{1}{(T')^2} \text{ is positive}\right)$$

which shows that we should use the third form of the general integral which is

$$z = + .354 + 1.33\, e^{-\frac{t}{165}} - 1.27\, e^{-\frac{t}{15.65}}$$

$$s = -.00805\, e^{-\frac{t}{165}} + .0810\, e^{-\frac{t}{15.65}}$$

which values give the following table:

| Seconds $t =$ | 0 | 50 | 100 | 150 | 200 |
|---|---|---|---|---|---|
| feet | $z = +.41$ | $+1.282$ | $+1.079$ | $+.881$ | $+.746$ |
| feet per sec. | $s = +.073$ | $-.00261$ | $-.0044$ | $-.0032$ | $-.0024$ |

The time of the highest elevation is determined with $s = o$ from equation $o = -.00805\, e^{-\frac{t_1}{165}} + .0810\, e^{-\frac{t_1}{15.65}}$ to $t_1 = 40$ seconds and $z$ max. follows:

$$z \max = + .354 + 1.33\, e^{-\frac{40}{165}} - 1.27\, e^{-\frac{40}{15.65}} = 1.30 \text{ ft.}$$

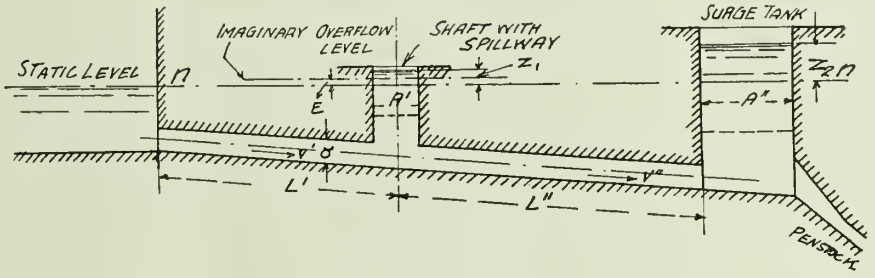


**Fig. 10.**

$$y = (R_1\, e^{+\frac{t}{T_2}} + R_2\, e^{-\frac{t}{T_2}})\, e^{-\frac{t}{2T_0'}}$$

Further, $m = \dfrac{-E}{\frac{A}{k}\frac{T_0}{T^2} + 1} = -.354$ feet

$T_0' = 14.3$ sec. $\quad T' = 51$ sec. $\quad T_2 = 34.6$ sec., so that

$$z = + .354 + [R_1\, e^{+\frac{t}{34.6}} + R_2\, e^{-\frac{t}{34.6}}]\, e^{-\frac{t}{28.6}}$$

$$= + .354 + R_1\, e^{-\frac{t}{165}} + R_2\, e^{-\frac{t}{15.65}}$$

$$s = \frac{dz}{dt} = -\frac{R_1}{165}\, e^{-\frac{t}{165}} - \frac{R_2}{15.65}\, e^{-\frac{t}{15.65}} \quad \text{and for } t = o$$

$$+ .410 = + .354 + R_1 + R_2; \quad .073 = -\frac{R_1}{165} - \frac{R_2}{15.65};$$

therefore, $R_1 = 1.33$ feet and $R_2 = -1.27$ feet

$$z = + .354 + 1.33\, e^{-\frac{t}{165}} - 1.27\, e^{-\frac{t}{15.65}}$$

This water level corresponds to an overflow quantity of 300 cubic feet per second. For the determination of the ideal overflow height and the factor of proportionality, we used the maximum overflow quantity of 282 cubic feet per second. It is therefore shown that the assumption made that $u = 0.7$ is correct. The duration of the overflow period follows from

$$.41 = + .354 + 1.33\, e^{-\frac{t}{165}} - 1.27\, e^{-\frac{t}{15.65}}$$

and is $t_x = 520$ seconds.

And therefore the velocity $s_x$ with which the water level reaches the ideal overflow elevation is:

$$s_x = -.00805\, e^{-\frac{520}{165}} + .0810\, e^{-\frac{520}{15.65}} = -.000346 \frac{\text{feet}}{\text{sec.}}$$

With these initial values the movement in the third period may be determined. Because we assumed that the spillway crest is at the elevation of the static level $n-n$, the overflow period goes further but with smaller fluctuations of water level and overflow quantities.

If the spillway is built not in the surge tank, but in the main conduit, the principal equation requires a supplement. According to Fig. 10, a shaft at the distance $L'$ from the beginning of the conduit is driven down to the conduit with a section $A'$, through which water from the conduit goes over a spillway whose ideal overflow crest is as before the distance $E$ from the static level $n-n$. At the distance $L''$ is the surge tank with the

section $A''$. In the hydraulic equilibrium with $Q_1$ cubic feet per second discharge through the penstocks, the water surface in the overflow shaft will be below the static level $n-n$ by the distance $h_1' = n'.v_1$; in the surge tank by $h_1'' = (n' + n'').v_1$. $v_1$ is the velocity in the conduit of the area $a$ which corresponds to the discharge $Q_1$. $n'$ and $n''$ are the function coefficients corresponding to the distances $L'$ and $L''$. For flow fluctuations the distance $z_1$ and $z_2$ from the static level $n-n$ at the velocities $v_1'$ and $v_1''$ and the water surfaces in shaft and surge tank will have different levels and there will be overflow in the shaft as soon as the water surface reaches the spillway crest. It is sufficient to carry through the solution of the principal equation for the latter period and to handle the first period as a special case of it.

For the purpose of simplifying the work, we consider only the case of sudden total shut-down. For both parts of the conduit, we get the following movement equations:

$$\frac{L'\,dv_1'}{g\,dt} + z_1 + n'.v_1' = o$$

The motion equations and the second equation for continuity reach their values before as well as after the overflow on the spillway. The second equation for continuity is correct for $k = o$, for periods without any overflow. If we consider that $s_1 = \frac{dz_1}{dt}$ and $s_2 = \frac{dz_2}{dt}$ we may, with the aid of the continuity equations, eliminate the velocities $v_1'$, $v_1''$ and their derivations. We get then two simultaneous differential equations of the second order, from which we eliminate again $z_1$ and its derivations, whereas for the determination of $z_2$ we get a linear differential equation of the fourth degree with constant coefficients, the integration of which does not involve any great difficulty.

In order to make the spillway especially efficient, we manage it so that the spillway crest lies below the static level $n-n$, and this in a distance which is equal to or a little less than $h_1' = n'.v_1$. In this case, after the shut-down is finished, a very rapid overflow will occur in the spillway and will continue, and finally there exists a constant flow over the spillway, whereby naturally as much flows through the upper conduit as goes away over the spillway and where the water level in the shaft, as well as in the surge tank lies at a distance under $n-n$, which corresponds to the hydraulic slope necessary for the flow through the upper conduit.

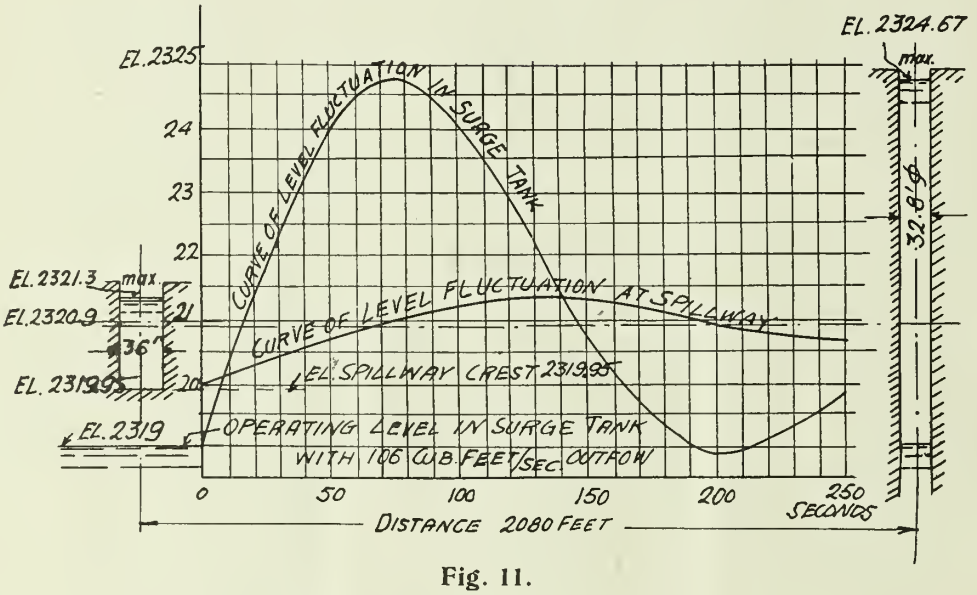


Fig. 11.

$$\frac{L''\,.dv_1''}{g.dt} + (z_2 - z_1) + n''\,v_1'' = o \qquad - \quad (86)$$

The equations which express continuity follow from the consideration that from the upper conduit in a unit of time $dt$, a quantity of water must flow into the shaft which is equal to the algebraic sum of—

1st—the quantity which flows away in the lower conduit, which is equal to $a.v_1''.dt$.

2nd—the simultaneous filling of the shaft with the quantity $A'.v_1'.dt$.

3rd—the simultaneous overflow quantity $k\,(z_1 - E)\,dt$. But the water quantity which flows through the lower conduit in the time $dt$ is also equal to the simultaneous filling $A''\,v_1''\,dt$ in the surge tank. The following two equations express, therefore, the continuity

$$a.v_1' = a.v_1'' + A'\,s_1 + k\,(z_1 - E)$$

and

$$av_1'' = A''s_2 \qquad - \quad - \quad - \quad - \quad (87)$$

If in this case $A'$ and the spillway widths are small enough with reference to the hydraulic slope, to keep the preceding fluctuations so near the static level of the water surface in the shaft, that the variation of the inflow to the shaft may be neglected, the problem becomes simplified, because the first motion equation drops out and in the first equation for continuity, the values $a.v_1'$ become constant and equal to $q$.

Considering that $s_1 = \frac{dz_1}{dt}$ and $s_2 = \frac{dz_2}{dt}$, we get from the first equation of continuity (87)

$$A'' \cdot \frac{dz_2}{dt} + A' \frac{dz_1}{dt} + k\,(z_1 + E) = q. \qquad (88)$$

and the second motion equation becomes zero.

$$\frac{L''}{g} A'' \frac{d^2 z_2}{dt^2} + r_2 A'' \frac{dz_2}{dt} + a\,(z_2 - z_1) = o \qquad (89)$$

From this last equation, $z_1$ and its derivative may be determined and substituted in equation 30c. For the solution of $z_2$ we get a linear differential equation of the third degree which may readily be integrated. The results of such an investigation are apparent in Fig. 11; we see that at the time of the highest water elevation in the surge tank the height of water flowing over the spillway is directly proportional to the height of the water in the surge tank above the static level, but the quantity of water is greater during the time the level in the surge tank is falling and continues until the level in the surge tank has reached its lowest elevation. The extent of the fluctuations is not only dependent upon the conduit, surge tank and spillway dimensions, but, naturally, also depends upon the area $A'$. The complete computation of the case would exceed the space available here.

### Investigation of the Influence of $h = u.v^2$.

Finally, a study may be added which permits a comparison between the results of the assumption already made (that $h = n.v$), and the results obtained if we use the usual formula $h = u.v^2 = u^1.\frac{Lv^2}{P.2g}$ where the friction coefficient $u^1$ is again assumed as constant ($P$ = hydraulic radius of the conduit). With this condition it is necessary to use the first original equation (15) in which we must, according to the direction of the flow in the conduit, be careful to use the right sign for $h$. If this makes the investigation somewhat complicated, it may be said that the differential equation, which we obtain introducing the value of $v$ in the continuity equation (17) is not only of the second (i.e., higher) order, but also of the second degree, so that difficulties occur in the determination of the general integral. This is especially the case if in the equation mentioned $q > o$ and variable.

Therefore, in the following we consider only for comparative purposes the case of a sudden shut-down of $Q$ and for the phase of the first surge. In order to get more generality, we first consider the surge tank section as variable with the height, which means that it is assumed as a given function of $z$. The motion equation is then:

$$\frac{L\,dv}{g\,dt} + z + h = o. \qquad - \quad - \qquad (15\text{ A}).$$

The equation of continuity is:

$$a.v = A.s \qquad - \quad - \quad - \qquad (17\text{ B})$$

If we multiply equation 15 A with $a.v.dt$, it follows, considering equation (17 B)

$$\frac{L.a}{g} v.dv + z A s\,dt + h.A.s.dt = o \qquad (90)$$

$A.s\,dt = A.dz = dV$ represents the change of the water volume $V$ in the time $dt$. It also represents, with given surge tank dimensions as function of $z$ or with reference to the graphical demonstration appearing further on, the function of a variable $x$ where $z = x - h_1$ and where $V$ is measured positive up from that level which lies below $n-n$. That is, for $z = -h_1$ or $x = o$; $V = o$.

With $h = \frac{u^1 L v^2}{P.2g}$ and $h_1 = \frac{u^1 L v_2^2}{P.2g}$ and multiplying (90) with $\frac{2g}{La}$ we get

$$d(v)^2 + \left[\frac{2g}{L.a} \cdot x + \frac{u^1}{P.a}(v^2 - v_2^2)\right] dV = o$$

If we say $v^2 = y \qquad v_2^2 = y_2 \qquad \frac{2g}{L.a} = a_1 \frac{u^1}{P.a} = \frac{1}{a_2}$ the total differential equation follows:

$$dy + \left[a_1 x + \frac{1}{a_2}(y - y_2)\right] dV = o$$ which integration can

be done introducing the integrating factor $\mu = e^{\frac{V}{a_2}}$

$$y - y_2 = -a_1 e^{-\frac{V}{a_2}} \int_0^V x\, e^{\frac{V}{a_2}}\, dV$$ and introducing the values for $y$, $a_1$, $a_2$.

$$v^2 = v_2^2 - \frac{2g}{L.a} e^{-\frac{u^1 V}{P.a}} \int_0^V x\, e^{\frac{u^1 V}{P.a}} . dV \qquad (91)$$

With help of this equation $v$ may be found as a function of $V$ and $x$, because by given relation of the volume $V$ to $x$ and vice versâ, in formula (91) $\int_0^V x\, e^{\frac{u^1 V}{P.a}} . dV$ may be determined by means of quadrature even if $V = F(x)$ is given only by a curve. The method is illustrated by the diagram shown in Fig. 12. The relation of the volume $V$ to $x$ is shown by a curve, the abscissæ of which represent the values of $V$ and the ordinates the values of $x$. The construction of the $x$ curve $I - I$ may be accomplished, based on the given dimensions of the surge tank. $V = o$ corresponds, as said before, to the operating level for a flow of $Q_1$ cubic feet per second. With the same abscissæ, we construct a curve II — II, the ordinates of which are given by the values $e^{+\frac{u^1 V}{P.a}}$ and another curve III — III with the ordinates $x.e^{\frac{u^1 V}{P.a}}$ and then the integral curve IV — IV. If we construct a further curve V — V, the ordinates of which we get if we multiply the ordinate values of the integral curve by $\frac{2g}{L.a} e^{\frac{u^1 V}{P.a}}$ and if we construct at the distance $(v_2^2)$ a

line parallel to the axis of abscissæ we then see that the difference of the ordinates between the $(v_2^2)$ line and the curve V — V gives the values of $v^2$. Naturally, we must select suitable scales for all curves, that is to say, for their ordinates. The scale for the last curve is naturally the same as for $(v_2^2)$. The intersection of the last curve with the parallel $(v_2^2)$ determines the ordinate which gives the value of $V$ on the axis of abscissæ, for which $v = o$, that is, when the highest water elevation occurs. The corresponding ordinate of the $x$ curve gives the height of the highest elevation above the operating level.

If $A$ is constant: $V = A.x$ and equation 91 becomes

$$v^2 = v_2^2 - \frac{2g.A}{L.a} \cdot e^{-u^1 \frac{A.x}{P.a}} \int_0^x x\, e^{u^1 \frac{A\,x}{P.a}}\, dx$$

If we introduce $\frac{2g}{L} \cdot \frac{A}{a} = \frac{1}{B^2}$; $u^1 \frac{A}{P.a} = \frac{1}{D}$ we get

$$v^2 = v_2^2 - \frac{e^{-\frac{x}{D}}}{B^2} \int_0^x x\, e^{\frac{x}{D}}\, dx$$

$$v^2 = v_2^2 + \frac{D^2}{B^2}\left(1 - \frac{x}{D} - e^{-\frac{x}{D}}\right) \qquad (92)$$

And for the computation of the maximum value $X$ of $x$ $(v = o)$

$$\frac{v_2^2 . B^2}{D^2} + 1 = \frac{X}{D} + e^{-\frac{X}{D}} \qquad (93)$$

The results here obtained may now be used for comparative purposes with the results of the example of case $(A)$.

$v_2 = 6.63$ feet per sec. $\qquad \frac{v_2^2}{2g} = .68$ feet

$h_1 = 9.6$ feet $= \frac{u^1}{P} \cdot 9050.0.68$

$$\frac{u^1}{P} = \frac{1}{641} \qquad \frac{1}{D} = \frac{u^1}{P} \cdot \frac{A}{a} = \frac{1}{9.6}$$

$$\frac{1}{B^2} = \frac{2gA}{L.a} = \frac{1}{2.094} \qquad B = 1.447 \text{ sec.}$$

$$\frac{v_2^2 B^2}{D^2} + 1 = \left(\frac{6.63^2 \times 1.447^2}{9.6^2}\right) + 1 = 2.00$$

Therefore

$$\frac{\frac{X}{D}}{\frac{X}{D} + e^{-\frac{X}{D}}} = 2.00$$

$$\frac{X}{D} = 1.86 \qquad X = 17.81 \text{ feet}$$

In case $(A)$, where we assume the same conditions with $h_1 = n.v$ in which $n = 1.445$ seconds, we got $z$ max $= 6.55'$ and therefore the highest elevation above the initial level $z$ max $+ h_1 = 6.55 + 9.6 = 16.15'$. This

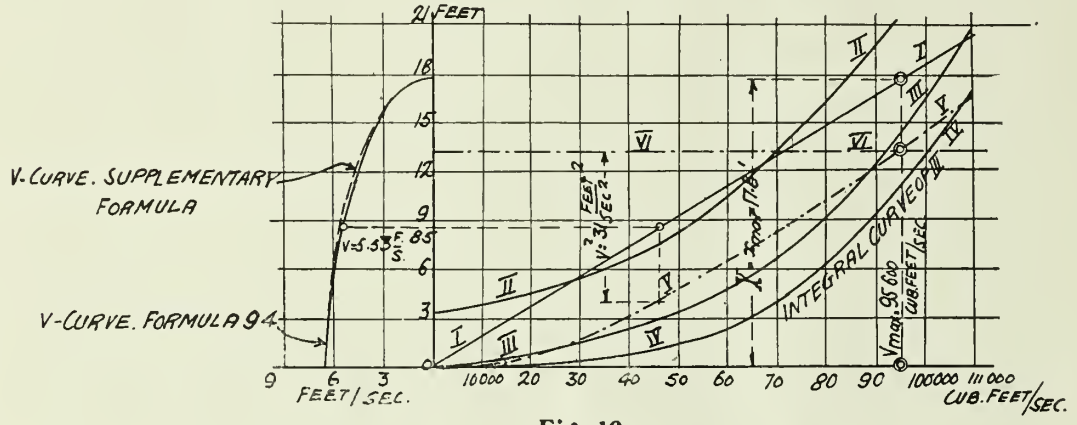


Fig. 12.

value is smaller than the value which we computed for $X$ by 1.66 feet, and it is necessary for us to correct the formulæ obtained under the assumption that $h = n.v$, in order to get as near as possible to the new results. For this purpose, we assume that for the expression of the elevation $x$ above the initial level the formula

$$\left.\begin{aligned} x = h_1 + z = h_1 + R.e^{-\frac{t}{2T_0}} \sin\left(\beta + \frac{t}{T_1}\right) \\ \frac{dx}{dt} = s = \frac{R}{T} . e^{-\frac{t}{2T_0}} \sin\left(2\gamma - \beta - \frac{t}{T_1}\right) \end{aligned}\right\} \qquad (94)$$

with $tg\,\gamma = \frac{2T_0}{T_1}$

and therefore

$$\frac{1}{T^2} = \frac{1}{T_1^2} + \frac{1}{(2T_0)^2}$$

$$\frac{d^2x}{dt^2} = -\frac{R}{T^2} e^{-\frac{t}{2T_0}} \sin\left(2\gamma - \beta - \frac{t}{T_1}\right) \qquad (95)$$

should be used, where the values $R$ and $\beta$ are again to be determined from the initial conditions, that is to say,

where $t = o$, $x = o$ and $z = h_1$, $\frac{dx}{dt} = c_1$, but the values

$T_0$ and $T$ and $T_1$ must be determined, not as before from the given dimension, but from the condition that $X$ becomes the maximum value of the elevation above the initial elevation, that is, for $x = X$ the derivative $\frac{dx}{dt}$ must become zero, and as before for $x = o$, the second derivative $\frac{d^2x}{dt^2}$ becomes zero. From the latter condition follows:

$2\gamma - \beta = 180° = \pi$ therefore $\gamma - \beta = 180 - \gamma$
$\beta = 2\gamma - 180°$ and for $t = o$ we get

$$o = h_1 - R \cdot \sin 2\gamma$$

$$c_1 = \frac{R}{2T_0} \sin\gamma \sqrt{\left(\frac{2T_0}{T_1}\right)^2 + 1}$$

because $\frac{1}{T} = \frac{1}{2T_0} \sqrt{\left(\frac{2T_0}{T_1}\right)^2 + 1}$. See equation (38).

The time $t_x$ when $x = x_{max} = X$ $\left(\frac{dx}{dt} = o\right)$ is computed from

$$\frac{t_x}{T_1} = \gamma - \beta = \pi - \gamma. \qquad \text{Therefore}$$

$$X = h_1 + R\, e^{-\frac{T_1}{2T_0} \cdot \frac{t_x}{T_1}} \sin(\beta + \gamma - \beta)$$

and, as $\frac{T_1}{2T_0} = ctg\ \gamma$

$$X = h_1 + R\, e^{-(\pi - \gamma)\, ctg\, \gamma} \cdot \sin\gamma \qquad (96)$$

and we can first determine $\gamma$ with

$$R = \frac{h_1}{\sin 2\gamma}; \qquad \cos\gamma \cdot e^{+(\pi - \gamma)\, cotg\, \gamma} = \frac{1}{2\left(\frac{X}{h_1} - 1\right)}$$

Is $\gamma$ with this equation computed, we have for the other values

$$R = \frac{h_1}{\sin 2\gamma}; \qquad 2T_0 = \frac{R}{c_1} \sqrt{tg^2\gamma + 1} = \frac{R}{C_1 \cdot \cos\gamma};$$

$$T_1 = \frac{2T_0}{tg\,\gamma}; \qquad \frac{1}{T} = \sqrt{\frac{1}{(2T_0)^2} + \frac{1}{T_1^2}}.$$

These values may then be introduced in the formulas for

$$x \text{ and } \frac{dx}{dt} = s \qquad - \quad - \quad - \qquad (94)$$

If the movement so determined must be similar to that which occurs if $h = u^1 \frac{L}{P} \cdot \frac{v^2}{2g}$, then the velocities $v = s \frac{A}{a}$, which we get for the same elevation $x$ in both cases, must have the same values. If these values are not exactly equal, but if we get approximately equal values it depends on the degree of approximation desired whether we use the supplementary formula (92) or not.

In order to investigate this case, we proceed to compute the example.

With $X = 17.81$ feet $h_1 = 9.6$ feet we get

$$\cos\gamma\, e^{+(\pi - \gamma)\, cotg\, \gamma} = \frac{1}{2\left(\frac{X}{h_1} - 1\right)} = .581 \quad \gamma = 71° 50'$$

$\beta = 2\gamma - 180° = -36° 20'$; $\quad tg\,\gamma = \frac{2T_0}{T_1} = 3.047$;

$$R = \frac{h_1}{\sin 2\gamma} = 16.2 \text{ feet with } c_1 = 0.098 \frac{\text{feet}}{\text{sec.}}$$

$$2T_0 = \frac{R}{c_1} \sin\gamma \sqrt{tg^2\gamma + 1} = \frac{R}{c_1} tg\,\gamma = 500 \text{ sec.}$$

$T_1 = 164.3$ sec. $T = 155.9$ sec. $t_x\ (x = X) = 310$ sec.

and therefore $x = 9.6 + 16.2\, e^{-\frac{t}{500}} \sin\left(\frac{t}{164.3} - 36°\right)$

$$s = \frac{16.2}{155.9} e^{-\frac{t}{500}} \sin\left(71° 50' + \frac{t}{164.3}\right); \quad v = \frac{s.A}{a} = s \times \frac{5380}{80}$$

With these formulæ, we may compute the relative values of $x$, $s$ and $v$ for the different values of $t$. From the supplementary formula (92)

$$v^2 = v_2^2 + \frac{D^2}{B^2}\left(1 - \frac{x}{D} - e^{-\frac{x}{D}}\right) \qquad - \qquad (92)$$

we may compute the values of $v$ for the same values of $x$ or they may be determined graphically from Fig. 12. They correspond to the assumption that $h = u^1 \frac{L}{P} \cdot \frac{v^2}{2g}$ and we get the following table of values:

| t Sec. | o | 50 | 100 | 150 | 200 | 250 | 300 | 310 |
|---|---|---|---|---|---|---|---|---|
| x feet | 0 | 4.06 | 9.21 | 12.9 | 15.7 | 17.4 | 17.8 | 17.81 |
| s feet/sec. | 098 | .092 | .082 | .0623 | .0425 | .023 | .00328 | .00 |
| and from that | | | | | | | | |
| v feet/sec. | 6.63 | 6.25 | 5.49 | 4.23 | 2.92 | 1.51 | .20 | .00 |
| But from Formula (92) | | | | | | | | |
| v feet/sec. | 6.63 | 6.3 | 5.38 | 4.05 | 2.83 | 1.35 | .066 | .00 |

If we plot in the rectangular co-ordinate system (Fig. 12) the two values of $v$ as ordinates with the values of $x$

as corresponding abscissæ, then we see from the curves thus obtained that we may use the supplementary formula with sufficient accuracy for the determination of the movement. If we compare the values of $T_0$ and $T$ as they have been found for both assumptions of $h$, that is to say, for $T_0$ the values 194.5 and 250.0 and for $T$ the values 137.5 and 155.9, then we see that the second value of $T_0$ is 1.28 times as large as the first, the second value of $T$ is $1.13 = \sqrt{1.28}$ times as large as its first value. The first values have been found from the formulæ $T_0 = \frac{L}{n.g}$ and $T = \sqrt{\frac{L}{g} \cdot \frac{A}{a}}$ and from this we see the importance of the supplementary formula (92). It is evolved from the principal equation 15 if we introduce instead of $L$, a distance $L^*$ which is 28% larger than $L$ (provided that the value $n$ is the same according to the relation $h_1 = n.v_1$) then with the distance $L^*$, we compute the values $T$ and $T_0$. It is evident that with the introduction of the formula of resistance $h_1 = n.v$, the influence of the friction has been introduced in the equation as too large. We have, therefore, assumed in the supplementary formula the working capacity and, therefore, also the mass of the conduit volume, larger than they really are, but obtained the right value of the elevation through the supplementary formula. The friction height which corresponds to the elevation $X = 17.81$ feet can be obtained by computing the work balance, according to the former scheme, if we introduce the actual length of the conduit.

$G$ becomes $5{,}380 \times 17.81 = 2{,}994$ tons;

$y_1 = \frac{1}{2}\ 17.81 = 8.9$ feet.

| USEFUL WORK IN FEET TONS | | USED WORK |
|---|---|---|
| 15,800 | KINETIC ENERGY OF CONDUIT VOLUME | |
| 28,740 | POTENTIAL ENERGY OF G = 2994t FOR h = 9·6 | |
| | LIFTING WORK OF 2990t FOR 8·9' | 26,650 \| TOTAL |
| TOTAL 44,540 | FRICTION WORK | 17,890 / 44,540 |

The total friction work of 17,900 foot-tons corresponds, therefore, to an average friction height with respect to the weight of the water of

$$h_{\text{average}} = \frac{17{,}900}{2{,}994} = .60\ h_1$$

For case $(A)$ we found $h_{\text{average}} = .755\ h_1$ as the average value which corresponds to an elevation of only 16.75 feet actually. The maximum elevation above the initial level will be larger than 16.75 feet and smaller than 17.81 feet, therefore, $0.60\ h_1 < h_{\text{average}} < .755\ h_1$. The true value will be apparent if observation results in constructions of larger dimensions than can be obtained. We may remark also that the above determined values of $T$ and $T_0$ in the supplementary formula are only correct for the first surge from $x = o$ to $x = X$. We may carry the mathematical investigation further and find the values for the drop from the first maximum elevation to the first minimum level. Other values for $T$ and $T_0$ must be introduced, which correspond to smaller values of $L^*$. This repeats itself for the further phases of increase and decrease until finally $L^*$ becomes nearly equal to $L$. This corresponds to the decrease of the velocity fluctuations during the movement and also with the decrease of the influence of friction.

## Conclusions.

We may summarize our results as follows:

1. If the conduit ends in a pond, from which the penstock goes directly to the turbine, and if the area of the pond surface $A > 150.m.a$ where $a$ = the conduit area in square feet and $m$ the number of miles conduit length: we need not expect any periodical fluctuations of the outflow, even if the outflow varies during the observed time. If $A < 150\ m\ a$ as is the case with artificial surge tanks, such periodical fluctuations may occur.

2. The maximum elevation above the initial level (operation level for full outflow) has approximately the same ratio for a short time for a sudden as well as for a gradual close of the whole outflow.

3. The value of the maximum elevation above the initial level may best be computed by means of the work balance, where for the average friction height 0.7 of the value of that height $h_1$ must be introduced, which corresponds to the drop of the water surface in the surge tank below the level of the water surface before the conduit for full outflow.

4. The extent of drop below the static level for full opening is approximately equal to the rise above computed.

5. For the determination of the oscillations during and after completed shut-down, we may use for a fair degree of approximation the formulæ, computations and demonstration methods which correspond to the analysis above developed on the theory of damped oscillations. The values of $T$ and $T_0$ and therefore also $T_1$ may be determined if investigating the maximum elevations, as shown in our last study.

6. For operation, which results in a periodical variation of the outflow, resonance features may occur.

7. The maximum elevations may be decreased by construction of spillways in the surge tank or in the conduit.

The theory furnishes a good explanation concerning the manner and the relative value of the operations close enough for most practical purposes. But a high degree of accuracy will be accomplished only where experiments furnish much more exact values for friction in conduits under different conditions.

## Appendix A—Surge Tank With Variable Cross-section.

Assuming a quadratical relation between friction-head and conduit velocity the following relation was obtained. (See Formula 91.)

$$v^2 = v_2^2 - \frac{2g}{L.a}\, e^{-\frac{u^1 V}{P.a}} \int_0^V x\, e^{+\frac{u^1 V}{P.a}}\, dV \qquad (91)$$

The maximum surge is obtained by making $v^2 = o$ and solving the latter equation for $X$. The previously described graphical method may be applied without further comment to surge tanks of variable cross-section with the exception that the $X$ curve is no longer a linear function of $V$. For this reason and because of complex integration due to the insertion of a complicated function the graphical method is more appropriate than the analytical.

In the following example this method is applied to a surge tank, circular in section, which has a sectional area of 5,380 sq. ft. at the operating level, as was assumed before. However, in this case the sectional area increases in direct proportion to the height in such a ratio that at a height of 19.7 ft. the sectional area is 6,450 sq.

ft. (See Fig. 14.) All other data for the problem is the same as assumed in the main article.

$L$ = 9,050 ft.
$a$ = 80 sq. ft.
$p$ = 32.8 ft.
$Q$ = 530 cu. ft. per sec.
$v_2$ = 6.63 ft. per sec.
$P$ = 2.44
$h_1$ = 9.6 ft.
$u'$ = .0038

$$\frac{u'}{P.a} = \frac{1}{51{,}300} = \frac{1}{a_2}$$

$$6.63^2 = 44.0 = \frac{64.4}{9050.80} e^{\frac{V}{51{,}300}} \int_0^V x\, e^{\frac{V}{51{,}300}}\, dV$$

With this graphical method (Fig. 13) a maximum surge of only 17.0 ft. is obtained, whereas in the former example, with a surge tank of a constant sectional area

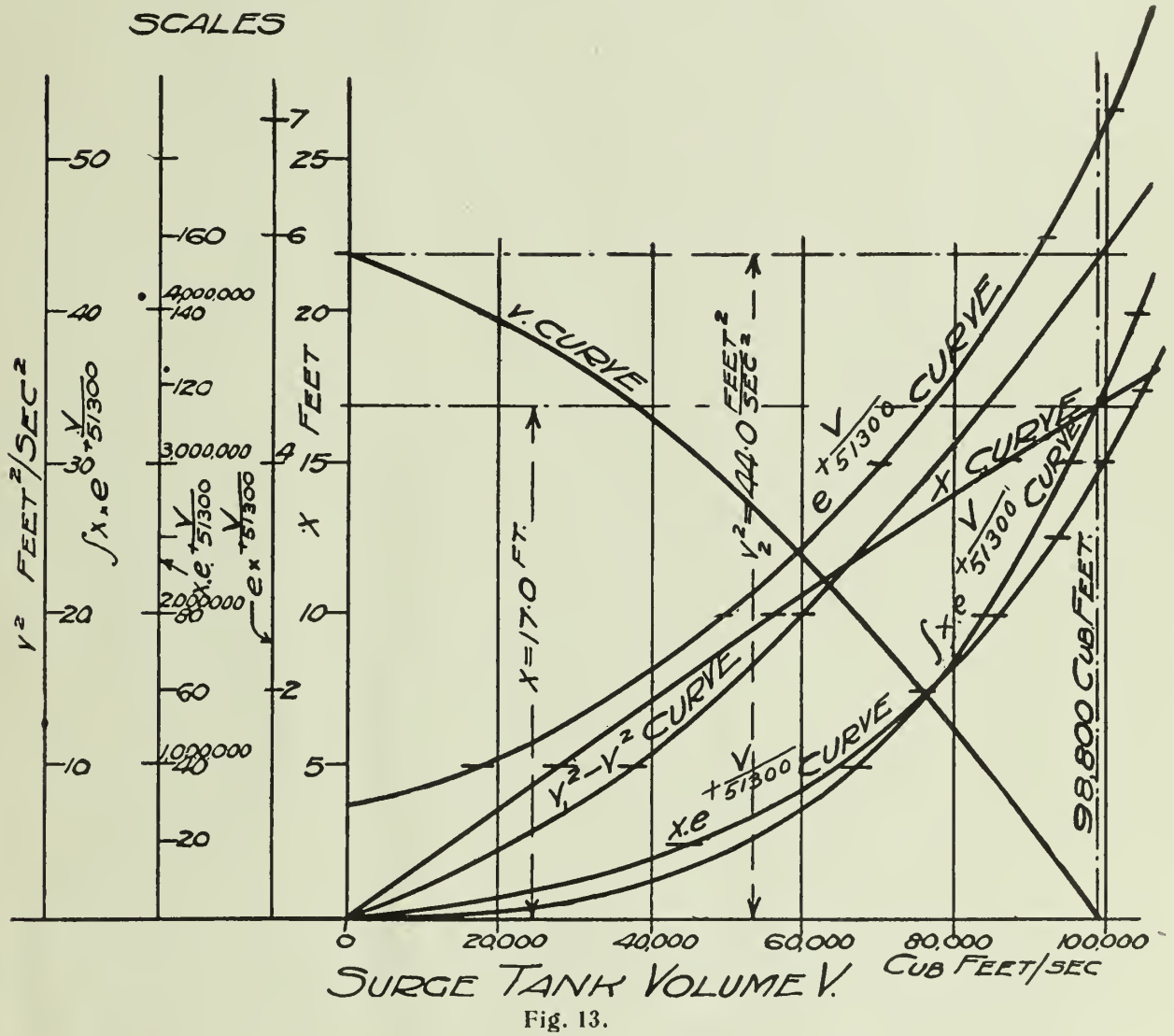


Fig. 13.

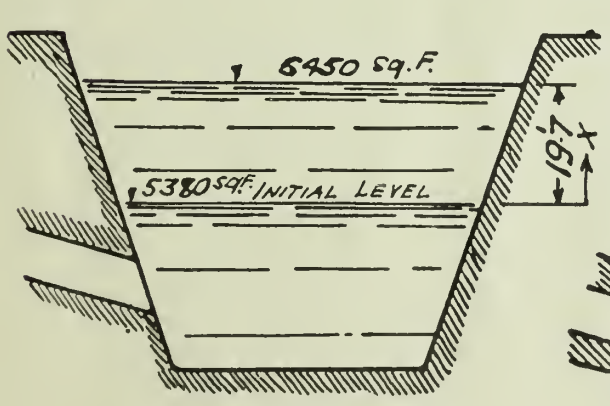


Fig. 14.

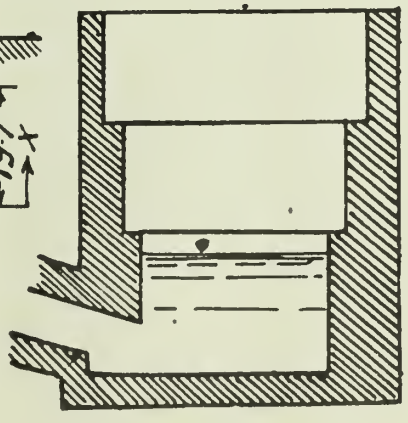

Fig. 15.

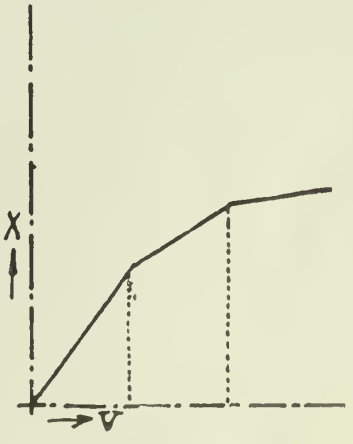


Fig. 16.

of 5,380 sq. ft. a surge of 17.8 ft. resulted. But the volume of water lifted is, in this case, somewhat greater, that is, 98,800 cu. ft., as compared to but 95,600 cu. ft. in the former case.

In the same easy way the graphical method may be applied to a surge tank in which the cross-section varies irregularly with the height. For instance, as the Albula plant of the city of Zurich. (Fig. 15.) In this a break occurs at the points on the $x$ curve where the ratio between sectional area and height changes. (See Fig. 16.)

## Appendix B—Simplified and Approximate Formula for the Determination of Surge Tanks With Constant Cross-Section.

By A. Strickler.

The designing engineer must often consider several solutions for one problem and from these select the best for the actual work. For these preliminary investigations he is in need of a simple slide-rule formula. In the following such a formula will be derived for the required area of a surge tank with given maximum surge.

As initial formula, equation 51 may be used for a sud-

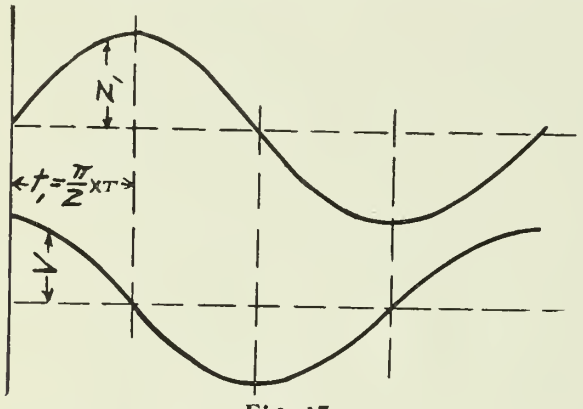


Fig. 17.

den full shut-down.

$$\frac{M \cdot v_1^2}{2} + G \cdot h_1 = G \cdot y_1 + A^1 \quad \cdot \quad (51)$$

The notation being the same as before, the average friction head in the surge tank for the first surge may be called $h_a$. Equation 51 can now be written:

$$-\frac{w \cdot a \cdot L}{2g} \cdot v_1^2 + G\left(\frac{y_{max}}{2} - h_1\right) + G \cdot h_a = o$$

$$-\frac{a \cdot L \cdot v_1^2}{2g} + \frac{A \cdot y_{max}^2}{2} - A \cdot y_{max} \cdot h_1 + A \cdot y_{max} \cdot h_a = o \quad (51a)$$

since $h = \frac{u^1 \cdot L}{P \cdot 2g} \cdot v_1^2$; $\qquad h_a = \frac{1}{t_1} \int_0^{t_1} h \cdot dt$

$$= \frac{1}{t_1} \int_0^{t_1} \frac{u^1 \cdot L \, v_1^2}{P \cdot 2g} \cdot dt \quad (51b)$$

$t_1$ = time from the beginning of the surge to the moment of maximum elevation. In order to integrate equation 51b, $v$ must be known as function of $t$. The simplifying assumption is now made that the first part of the $v$ curve is a regular cosine curve. This would be true if the water level fluctuations were an undamped harmonic of the form

$z = C \cdot \sin\left(\frac{t}{T}\right)$ (see Fig. 17).

Making the above assumption

$$v = \frac{s \cdot A}{a} = \frac{A}{a} \cdot \frac{dz}{dt} = \frac{A}{a} \cdot \frac{C}{T} \cdot \cos\left(\frac{t}{T}\right)$$

and also $\quad v_1 = \frac{A}{a} \cdot \frac{C}{T} \cdot \cos\frac{o}{T} = \frac{A}{a} \cdot \frac{C}{T}$

$$v = v_1 \cos\frac{t}{T}; \qquad h = h_1 \cdot \cos^2\left(\frac{t}{T}\right)$$

Substituting these values in equation (51b) there results:

$$h_a = \frac{1}{t_1} \int_0^{t_1} h_1 \cdot \cos^2\left(\frac{t}{T}\right) dt = \frac{h_1}{\frac{\pi}{2} \cdot T} \int_0^{\frac{\pi}{2} \cdot T} \cos^2\left(\frac{t}{T}\right) dt \quad (51c)$$

$$= \frac{h_1}{\frac{\pi}{2} \cdot T} \times \frac{\pi \cdot T}{4} = \frac{h_1}{2} \qquad \therefore \qquad h_a = \frac{h_1}{2}.$$

The average friction head in the tank for the time up to the maximum surge is therefore equal, under the above mentioned assumption, to half the friction head which exists in the conduit during operation. This value of $h_a$ introduced into equation 51a (work balance) gives:

$$-\frac{a \cdot L}{2g} \cdot v_1^2 + \frac{A}{2} \cdot y_{max}^2 - A \cdot y_{max} \cdot h_1 + \frac{A \cdot y_{max} \cdot h_1}{2} = o$$

$$\text{or } A = a \cdot \frac{\frac{L}{g} \cdot v_1^2}{y_{max}(y_{max} - h_1)} = \frac{a \frac{L \cdot v_1^2}{g}}{(z_{max} + h_1) z_{max}}$$

$$\therefore A = \frac{a \cdot v_1^2}{2g} \cdot \frac{2L}{(z_{max} + h_1) z_{max}}.$$

This formula gives results somewhat too large (1% to 3%) but is exact enough for preliminary computations.

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## Bibliography.

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